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ON THE COEFFICIENTS OF MEROMORPHIC UNIVALENT FUNCTIONS

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ABSTRACT. Let $f \in \Sigma$, the class of all analytic univalent functions defined in $\gamma = \{z: |z| > 1\}$. For $f, g \in \Sigma$ define h in γ by $h(z) = f(z)^{1-\alpha} g(z)^{\alpha}$, $0 < \alpha < 1$. If $h(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$, it is shown that $\sum_{n=1}^{\infty} n|c_n|^2 < \infty$. This result is used to show that if B_α denotes the class of all meromorphic Bazilevič functions of type α and $f \in B_\alpha$ with $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$, then $na_n = O(1)$ as $n \to \infty$, the result being best possible.

Denote by Σ the class of all analytic univalent functions f defined in $\gamma = \{z: |z| > 1\}$ by the power series $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$. Then, as is well known,

$$\sum_{n=1}^{\infty} n|a_n|^2 \leq 1,$$

which implies that $a_n = o(1)n^{-1/2}$ as $n \to \infty$. Clunie and Pommerenke [3] have shown that this result is not best possible and that $a_n = O(1)n^{-1/2-1/300}$ as $n \to \infty$. In the other direction, Clunie [1] has given an example of a function $f \in \Sigma$ satisfying $na_n > n^{\delta}$ for $\delta > 0$, and infinitely many n. The correct index of n, the so-called Clunie constant, is unknown.

For various subclasses of Σ the situation is much better. For example, if Σ^* denotes the class of all meromorphic starlike functions defined in γ, then Clunie [2] has shown that $a_n = O(1)n^{-1}$ as $n \to \infty$, and that the index of n is best possible. The same result has been proved by Pommerenke [6] for meromorphic close-to-convex functions. In fact Pommerenke was able to show that $a_n = O(1)n^{-1}$ holds for functions $f \in \Sigma$ satisfying $\mathrm{Re}(zf'(z)/g(z)) > 0$ for $z \in \gamma$ with $g \in \Sigma$, $g(z) \neq 0$.

In [7], the following subclasses of Σ were introduced. Let $f \in \Sigma$, $g \in \Sigma^*$ and for real α let $\mathrm{Re}(zf'(z)/f(z)^{1-\alpha} g(z)^{\alpha}) > 0$ for $z \in \gamma$. Then f defined in this way is called a meromorphic Bazilevič function of type α. We denote these classes by $B(\alpha)$, and note that $B(0)$ and $B(1)$ are the classes of meromorphic starlike and meromorphic close-to-convex functions

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respectively. In [7], the coefficient problem for $B(\alpha)$ was considered, and it was shown that if $f \in B(\alpha)$, with $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$, $0 \leq \alpha \leq 1$, and f and g have no zeros in γ, then $na_n = O(1)(\log n)^{1/2}$ as $n \to \infty$. We shall show that the logarithmic factor can be removed, giving the best possible estimate $na_n = O(1)$ as $n \to \infty$. We prove in fact:

Theorem 1. *Let $f, g \in \Sigma$, with $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$, and suppose that both f and g have no zeros in γ. Then if for $0 \leq \alpha \leq 1$*

$$\text{Re}\, \frac{zf'(z)}{f(z)^{1-\alpha} g(z)^{\alpha}} > 0 \tag{2}$$

for $z \in \gamma$, $na_n = O(1)$ as $n \to \infty$.

Note that the result is proved for a wider class than $B(\alpha)$, since g need not necessarily belong to Σ^*. We do however still require that f and g have no zeros in γ.

In order to prove Theorem 1, we require the following result, which is of interest in itself.

Theorem 2. *Let $f, g \in \Sigma$ with*

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n} \quad \textit{and} \quad g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}.$$

For $0 < \alpha < 1$, and for sufficiently large z, define the function h in γ by

$$h(z) = f(z)^{1-\alpha} g(z)^{\alpha} = z + \sum_{n=0}^{\infty} c_n z^{-n}.$$

Then $\sum_{n=1}^{\infty} n|c_n|^2 < \infty$.

We need the following area principle due to Grunsky [4].

Lemma. Let *the function k be defined for $z \in \gamma$ by $k(z) = \sum_{n=0}^{m} d_n z^{-n} + \sum_{n=-\infty}^{-1} d_n z^n$, and suppose that $d_m \neq 0$. If k is strongly m-valent in γ, that is, k assumes in γ no value more than m times, then*

$$\sum_{n=-\infty}^{-1} |n|\, |d_n|^2 \leq \sum_{n=1}^{m} n\, |d_n|^2.$$

Proof of Theorem 2. A simple limiting argument shows that in order to prove the result, it is sufficient, provided the bound obtained for $\sum_{n=1}^{\infty} n|c_n|^2$ is independent of α, to consider α to be rational.

Assume then, that $\alpha = p/q$ where p, q are positive integers with $p < q$. Since f and g have no zeros in γ, we can define F and G in Σ by

$F(z) = f(z^q)^{1/q}$ and $G(z) = g(z^q)^{1/q}$. Further let $\mathfrak{F}$ and $\mathfrak{G}$ be defined for $z \in \gamma$ by $\mathfrak{F}(z) = F(z)^{q-p}$ and $\mathfrak{G}(z) = G(z)^p$ respectively. Then for $z \in \gamma$ we can write

$$\mathfrak{F}(z) = z^{q-p} + \sum_{n=1}^{\infty} A_n z^{-nq+q-p}, \qquad \mathfrak{G}(z) = z^p + \sum_{n=1}^{\infty} B_n z^{-nq+p}.$$

Clearly $\mathfrak{F}$ is strongly $q-p$-valent and $\mathfrak{G}$ strongly p-valent in γ. Thus by the Lemma,

$$\sum_{n=1}^{\infty} (nq+p-q)|A_n|^2 \le q-p, \tag{3}$$

and

$$\sum_{n=1}^{\infty} (nq-p)|B_n|^2 \le p. \tag{4}$$

Next define the function H in γ by $H(z) = \mathfrak{F}(z)\mathfrak{G}(z)$. Then for $z \in \gamma$ we can write $H(z) = z^q + \sum_{n=0}^{\infty} C_n z^{-nq}$. Now $H'(z) = \mathfrak{F}'(z)\mathfrak{G}(z) + \mathfrak{F}(z)\mathfrak{G}'(z)$, and so by the Schwarz inequality

$$|H'(z)|^2 \le (|\mathfrak{F}(z)|^2 + |\mathfrak{G}(z)|^2)(|\mathfrak{F}'(z)|^2 + |\mathfrak{G}'(z)|^2).$$

Let $\rho > 1$, and $M(\rho) = \max_{1<|z|\le\rho} (|\mathfrak{F}(z)|^2 + |\mathfrak{G}(z)|^2)$, and let $1 < \rho_0 < \rho$. Then

$$\int_{\rho_0}^{\rho}\int_0^{2\pi} |H'(re^{i\theta})|^2 r\,dr\,d\theta \le M(\rho)\int_{\rho_0}^{\rho}\int_0^{2\pi} (|\mathfrak{F}'(z)|^2 + |\mathfrak{G}'(z)|^2) r\,dr\,d\theta,$$

and so

$$q\rho^{2q} - \sum_{n=1}^{\infty} nq|C_n|^2\rho^{-2nq} - \left\{q\rho_0^{2q} - \sum_{n=1}^{\infty} nq|C_n|^2\rho_0^{-2nq}\right\}$$

$$\le M(\rho)\Bigg[(q-p)\rho^{2(q-p)} - \sum_{n=1}^{\infty}(nq-q+p)|A_n|^2\rho^{-2(nq-q+p)}$$

$$- \left\{(q-p)\rho_0^{2(q-p)} - \sum_{n=1}^{\infty}(nq-q+p)|A_n|^2\rho_0^{-2(nq-q+p)}\right\}$$

$$+ p\rho^{2p} - \sum_{n=1}^{\infty}(nq-p)|B_n|^2\rho^{-2(nq-p)}$$

$$- \left\{p\rho_0^{2p} - \sum_{n=1}^{\infty}(nq-p)|B_n|^2\rho_0^{-2(nq-p)}\right\}\Bigg].$$

Using (3) and (4) we see that letting $\rho_0 \to 1$

$$q\rho^{2q} - \sum_{n=1}^{\infty} nq|C_n|^2\rho^{-2nq} - \left\{q - \sum_{n=1}^{\infty} nq|C_n|^2\right\} \tag{5}$$
$$\leq M(\rho)[(q-p)\rho^{2(q-p)} + p\rho^{2p}].$$

Let $|f(z)| \leq K$ and $|g(z)| \leq K$ for $1 < |z| \leq 2$, and choose $\rho = 2^{1/q}$. Then $M(\rho) = M(2^{1/q}) \leq K^{2(q-p)/q} + K^{2p/q} \leq 2K^2$, since $K \geq 1$. Thus (5) gives

$$4q - \sum_{n=1}^{\infty} nq|C_n|^2 4^{-n} - \left(q - \sum_{n=1}^{\infty} nq|C_n|^2\right)$$
$$\leq 2K^2[(q-p)4^{(q-p)/q} + p4^{p/q}] \leq 16K^2 q,$$

and so

$$\sum_{n=1}^{\infty} n|C_n|^2(1 - 4^{-n}) \leq 16K^2 - 3,$$

which gives $\Sigma_{n=1}^{\infty} n|C_n|^2 \leq (4/3)(16K^2 - 3)$. Finally note that $H(z^{1/q}) = F(z^{1/q})^{q-p}$, $G(z^{1/q})^p = f(z)^{1-\alpha}g(z)^\alpha$, and so $C_n = c_n$ for $n \geq 1$, and the result follows.

It remains only to establish the result when f and g have a common zero in γ. Suppose then that $f(\zeta) = g(\zeta) = 0$ with $\zeta \in \gamma$. Then for $1 < |z| < (1 + |\zeta|)/2$, $1/K < |f(z)|$, $|g(z)| \leq K$ for some $K > 0$. Now

$$h'(z) = (1-\alpha)f'(z)f(z)^{-\alpha}g(z)^\alpha + \alpha g'(z)f(z)^{1-\alpha}g(z)^{\alpha-1},$$

and so for $1 < |z| \leq (1 + |\zeta|)/2$, $|h'(z)|^2 \leq K_1(|f'(z)|^2 + |g'(z)|^2)$, where K_1 is constant. Now let $\rho_0 > 1$; then as before,

$$\rho^2 - \sum_{n=1}^{\infty} n|c_n|^2\rho^{-2n} - \left\{\rho_0^2 - \sum_{n=1}^{\infty} n|c_n|^2\rho_0^{-2n}\right\}$$
$$\leq K_1\left[\rho^2 - \sum_{n=1}^{\infty} n|a_n|^2\rho^{-2n} - \left(\rho_0^2 - \sum_{n=1}^{\infty} n|a_n|^2\rho_0^{2n}\right)\right.$$
$$\left. + \rho^2 - \sum_{n=1}^{\infty} n|b_n|^2\rho^{-2n} - \left(\rho_0^2 - \sum_{n=1}^{\infty} n|b_n|^2\rho_0^{-2n}\right)\right].$$

Hence, letting $\rho_0 \to 1$, and using (1), we have

$$\sum_{n=1}^{\infty} n|c_n|^2(1 - \rho^{-2n}) \leq 1 + \rho^2(2K_1 - 1),$$

and so

$$\sum_{n=1}^{\infty} n|c_n|^2 \le \frac{\rho^2}{\rho^2-1}[1+\rho^2(2K_1-1)] \le K_2$$

on choosing $\rho = (1+|\zeta|)/2$. This proves Theorem 2.

Proof of Theorem 1. Since f and g have no zeros in γ, the function h, defined in γ by $h(z) = f(z)^{1-\alpha}g(z)^{\alpha}$ is analytic in γ. Thus from (2) we may write

$$zf'(z) = h(z)p(z), \tag{6}$$

where p is analytic in γ, $p(\infty) = 1$ and $\operatorname{Re} p(z) > 0$ for $z \in \gamma$. From (6) we thus have

$$z - \sum_{k=1}^{\infty} ka_k z^{-k} = \left(z + \sum_{k=0}^{\infty} c_k z^{-k}\right)\left(1 + \sum_{k=1}^{\infty} p_k z^{-k}\right),$$

and so for $n \ge 1$,

$$-na_n = p_{n+1} + \sum_{\nu=0}^{n-1} c_\nu p_{n-\nu} + c_n. \tag{7}$$

Cauchy's theorem gives for $k \ge 1$,

$$p_k = \frac{r^k}{\pi}\int_0^{2\pi} \operatorname{Re} p(re^{i\theta})e^{ik\theta}\,d\theta,$$

so that in (7) we have for $n \ge 1$,

$$-na_n = \frac{1}{\pi}\int_0^{2\pi} \operatorname{Re} p(re^{i\theta})\left(r^{n+1}e^{i(n+1)\theta} + \sum_{\nu=0}^{n-1} c_\nu r^{n-\nu}e^{i(n-\nu)\theta}\right)d\theta + c_n$$

$$= \frac{r^n}{\pi}\int_0^{2\pi} \operatorname{Re} p(re^{i\theta})\left(h(re^{i\theta}) - \sum_{\nu=n}^{\infty} c_\nu (re^{i\theta})^{-\nu}e^{in\theta}\right)d\theta + c_n.$$

Thus, since $\int_0^{2\pi} \operatorname{Re} p(re^{i\theta})\,d\theta = 2\pi$,

$$n|a_n| \le \frac{r^n}{\pi}\left|\int_0^{2\pi} \operatorname{Re}(re^{i\theta})h(re^{i\theta})e^{in\theta}d\theta\right| + 2r^n\sum_{\nu=n}^{\infty}|c_\nu|r^{-\nu} + |c_n|$$

$$\le \frac{r^n}{\pi}\left|\int_0^{2\pi} \operatorname{Re} p(re^{i\theta})h(re^{i\theta})e^{in\theta}d\theta\right| + 2r^n\left(\sum_{\nu=n}^{\infty}\nu|c_\nu|^2\right)^{1/2}\left(\sum_{\nu=n}^{\infty}\frac{1}{\nu r^{2\nu}}\right)^{1/2} + |c_n|$$

by the Schwarz inequality. Now since f and g have no zeros in γ, $|f(z)|$, $|g(z)| \le r+3$ [5], and so $|h(z)| \le r+3$. Also since by Theorem 2, $\sum_{\nu=1}^{\infty}\nu|c_\nu|^2 < \infty$, $|c_n| \le K$, and if $\epsilon_n^2 = \sum_{\nu=n}^{\infty}\nu|c_\nu|^2$, then $\epsilon_n^2 \le K$ and $\epsilon_n \to 0$

as $n \to \infty$. Thus

$$n|a_n| \leq 2(r+3)r^n + 2\epsilon_n(n(1-1/r^2))^{-1/2} + K,$$

and so for $n \geq 2$, choosing $r = (1-1/n)^{-1/2}$, we have $na_n = O(1)$ as $n \to \infty$.

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