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NONSELFADJOINT REPRESENTATIONS OF C^*-ALGEBRAS

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ABSTRACT. The following strengthening of a result of B. A. Barnes is proved: If ϕ is a topologically irreducible representation of a C^*-algebra $\mathfrak{A}$ on a Banach space such that $\phi(\mathfrak{A})$ contains a nonzero finite-rank operator, then ϕ is similar to an irreducible *-representation of $\mathfrak{A}$ (and is thus automatically continuous).

By an *operator algebra* on a Banach space $\mathfrak{X}$ we shall mean a (not necessarily closed) subalgebra of $\mathfrak{B}(\mathfrak{X})$, the algebra of all (bounded) operators on $\mathfrak{X}$. An operator algebra $\mathfrak{A}$ on $\mathfrak{X}$ is *transitive* (= irreducible) if the only (closed) subspaces of $\mathfrak{X}$ invariant under (every operator in) $\mathfrak{A}$ are $\{0\}$ and $\mathfrak{X}$; it is *strictly transitive* (= strictly irreducible) if $\{0\}$ and $\mathfrak{X}$ are the only linear manifolds invariant under it. If $\mathfrak{M}$ is a linear manifold invariant under an operator A or under an operator algebra $\mathfrak{A}$, then $A|\mathfrak{M}$ or $\mathfrak{A}|\mathfrak{M}$ denotes the appropriate restriction of the operator or the algebra to $\mathfrak{M}$. We shall also use the following notations: let B be an operator on $\mathfrak{X}$, $\mathfrak{D}$ a subset of $\mathfrak{X}$, and $\mathfrak{A}$ a set of operators on $\mathfrak{X}$; then $\mathfrak{A}B = \{AB: A \in \mathfrak{A}\}$, $\mathfrak{A}x = \{Ax: A \in \mathfrak{A}\}$, and $\mathfrak{A}\mathfrak{D} = \{Ax: A \in \mathfrak{A}, x \in \mathfrak{D}\}$.

Let $\mathfrak{A}$ be a C^*-algebra and let ϕ be a (topologically) irreducible representation of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}$. Barnes [1] proved that if $\mathfrak{A}/\phi^{-1}(0)$ contains a minimal one-sided ideal and if ϕ is continuous, then ϕ is similar to an irreducible (and hence, by Kadison's theorem [3], a strictly irreducible) *-representation of $\mathfrak{A}$. Our main result (Theorem 2 below) shows that the continuity of ϕ follows automatically from the other hypotheses.

We need the following elementary and purely algebraic lemma whose proof can be found, e.g., in [5].

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Lemma 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be operator algebras on the Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ respectively, and assume that $\mathfrak{B}$ is transitive and contains a rank-one operator E. Assume that ϕ is an algebra isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Then there exists a one-to-one linear transformation S from the (dense, $\mathfrak{B}$-invariant) linear manifold $\mathfrak{D} = \mathfrak{B}E\mathfrak{Y}$ into $\mathfrak{X}$ such that $AS = S\phi(A)$, on $\mathfrak{D}$, for all A in $\mathfrak{A}$.*

Lemma 2. *Let $\mathfrak{A}$, $\mathfrak{B}$, and ϕ be as in Lemma 1, and assume, in addition, that $\mathfrak{A}$ contains all the finite-rank operators on $\mathfrak{X}$. Then the transformation S of Lemma 1 is bicontinuous from $\mathfrak{Y}$ onto $\mathfrak{X}$.*

Proof. Since $\mathfrak{A}$ and $\mathfrak{B}$ are transitive, it follows from [5, Lemma 2 and Theorem 1] that S is closable. The hypothesis that $\mathfrak{A}$ contains all the finite-rank operators on $\mathfrak{X}$ together with the equation $ASx = S\phi(A)x$ for all A in $\mathfrak{A}$ and x in $E\mathfrak{Y}$ imply that $S\mathfrak{D} = \mathfrak{X}$ and, therefore, S^{-1} is a bounded operator by the closed graph theorem. The boundedness of S on $\mathfrak{D}$ is also easy to show. Let $\mathfrak{D}_1$ be the unit sphere of $\mathfrak{D}$; for each rank-one operator A in $\mathfrak{A}$, the transformation S is bounded on the one-dimensional space $\phi(A)\mathfrak{D}$ and hence $A(S\mathfrak{D}_1) = S\phi(A)\mathfrak{D}_1$ is a bounded set. Thus $f(S\mathfrak{D}_1)$ is bounded for every bounded linear functional on $\mathfrak{X}$; it follows that $S\mathfrak{D}_1$ is a bounded set.

Theorem 1. *Let $\mathfrak{A}$ be a C^*-algebra containing a minimal one-sided ideal and let ϕ be a faithful, irreducible representation of $\mathfrak{A}$ on a Banach space $\mathfrak{X}$. Then ϕ is similar to a faithful, irreducible *-representation of $\mathfrak{A}$. In particular, ϕ is automatically bicontinuous.*

Proof. We can assume that $\mathfrak{A}$ is a uniformly closed, selfadjoint algebra of operators on a Hilbert space $\mathfrak{H}$, and that $\mathfrak{A}$ contains a minimal (selfadjoint) projection $P \neq 0$, so that $\mathfrak{A}P$ is a minimal left ideal [6, p. 261]. Then $\phi(\mathfrak{A})\phi(P)$ is a minimal left ideal of $\phi(\mathfrak{A})$. Since $\phi(\mathfrak{A})$ is transitive, it follows that $\phi(P)$ is a rank-one idempotent in $\phi(\mathfrak{A})$. Thus Lemma 1 is applicable, and there exists a one-to-one linear transformation S from $\mathfrak{D} = \phi(\mathfrak{A})\phi(P)\mathfrak{X}$ into $\mathfrak{H}$ such that $AS = S\phi(A)$, on $\mathfrak{D}$, for all $A \in \mathfrak{A}$.

Now $S\mathfrak{D}$ is invariant under $\mathfrak{A}$. Thus the subspace $\mathfrak{M} = \overline{S\mathfrak{D}}$ of $\mathfrak{H}$ is invariant and therefore reducing for $\mathfrak{A}$. Let $\mathfrak{A}_1 = \mathfrak{A}|\mathfrak{M}$. The algebra $\mathfrak{A}_1$ is clearly *-isomorphic to $\mathfrak{A}$ (note that if $A|\mathfrak{M} = 0$ for A in $\mathfrak{A}$, then $S\phi(A)S^{-1} = 0$ on $S\mathfrak{D}$, hence $\phi(A) = 0$ and $A = 0$). It follows that $\mathfrak{A}_1$ is a uniformly closed selfadjoint subalgebra of $\mathfrak{B}(\mathfrak{M})$ which is a faithful *-representation of $\mathfrak{A}$ on $\mathfrak{M}$ (cf. [8, p. 5]).

Since $A|\mathfrak{M} = S\phi(A)S^{-1}$ on $S\mathfrak{D}$, the theorem will follow from Lemma 2 if we show that $\mathfrak{A}_1$ contains all the finite-rank operators on $\mathfrak{M}$. For F in $\mathfrak{A}$, if $\phi(F)$ has rank 1, so does $F|\mathfrak{M}$, because $F|S\mathfrak{D} = S\phi(F)S^{-1}$ has rank 1. Now $\phi(\mathfrak{A})$ contains rank-one operators in profusion : in fact for every nonzero x in $\mathfrak{D}$, there exists in $\phi(\mathfrak{A})$ a rank-one operator with range $\{x\}$ (by definition of $\mathfrak{D}$). It follows that for every nonzero y in $S\mathfrak{D}$, the algebra $\mathfrak{A}_1$ contains a rank-one operator F with the range $\{y\}$; hence it contains the rank-one projection $FF^*/\|FF^*\|$ with range $\{y\}$. Since $S\mathfrak{D}$ is dense in $\mathfrak{M}$, we conclude that $\mathfrak{A}_1$ contains the rank-one projection with range $\{y\}$ for every y in $\mathfrak{M}$. Thus every selfadjoint operator of finite rank is in $\mathfrak{A}_1$ and so is every arbitrary finite-rank operator.

(The bicontinuity of ϕ follows from the isometric character of *-isomorphisms [8, p. 5].)

Theorem 2. *Let ϕ be an irreducible representation of a C^*-algebra $\mathfrak{A}$ on the Banach space $\mathfrak{X}$. If $\mathfrak{A}/\phi^{-1}(0)$ contains a minimal one-sided ideal, then ϕ is similar to an irreducible *-representation of $\mathfrak{A}$. In particular, ϕ is continuous.*

Proof. With no loss of generality assume that both $\mathfrak{A}$ and $\phi(\mathfrak{A})$ have identities, because otherwise identities can be adjoined and homomorphisms extended in the usual fashion.

It suffices to show that $\phi^{-1}(0)$ is closed, for then $\mathfrak{A}/\phi^{-1}(0)$ would be a C^*-algebra [8, p. 43] and ϕ would induce an isomorphism ϕ_0 of $\mathfrak{A}/\phi^{-1}(0)$ onto $\phi(\mathfrak{A})$; by Theorem 1, ϕ_0 would be similar to a strictly irreducible *-representation of $\mathfrak{A}/\phi^{-1}(0)$ and therefore to one of $\mathfrak{A}$.

The proof of the closure of $\phi^{-1}(0)$ is very similar to the argument given in the proof of Rickart's theorem in [8, p. 162]: $\phi(\mathfrak{A})$ is transitive and contains nonzero finite-rank operators. It follows that $\phi(\mathfrak{A})$ is semisimple. (For otherwise the radical of $\phi(\mathfrak{A})$, as a two-sided ideal, contains all the finite-rank operators in $\phi(\mathfrak{A})$ and, in particular, finite-rank idempotents, but in a normed algebra the radical consists only of quasinilpotent elements.) Since $\phi(\mathfrak{A})$ has an identity, we have $\{0\} = \bigcap\{\mathfrak{I} : \mathfrak{I}$ maximal left ideal of $\phi(\mathfrak{A})\}$ (cf. [6, p. 55]). It follows that $\phi^{-1}(0) = \bigcap\{\phi^{-1}(\mathfrak{I})$: $\mathfrak{I}$ maximal left ideal of $\phi(\mathfrak{A})\}$; but every $\phi^{-1}(\mathfrak{I})$ is a maximal left ideal of $\mathfrak{A}$ and hence is closed in $\mathfrak{A}$. Thus $\phi^{-1}(0)$ is closed.

(The continuity of ϕ follows from that of *-representations of C^*-algebras.)

Remark. Using Lemma 2, a result of Barnes, and the recent theorem of

Lomonosov [4] gives the following result: If $\mathfrak{A}$ is a uniformly closed, transitive algebra of operators on a reflexive Banach space $\mathfrak{X}$ which contains a nonzero compact operator, then every isomorphism of $\mathfrak{A}$ onto a transitive algebra of operators on a Banach space $\mathfrak{Y}$ is spatial. To show this observe that, by Lomonosov's theorem, $\mathfrak{A}$ contains a compact operator with nonzero point spectrum. Since $\mathfrak{A}$ is uniformly closed, it contains nonzero finite-rank operators: the Riesz projections of a compact operator obtained by integration around isolated nonzero points of the spectrum have finite rank and belong to the uniformly closed algebra generated by the operator (see [7, p. 42]). By Barnes' theorem [2] $\mathfrak{A}$ contains all finite-rank operators on $\mathfrak{X}$. Thus the isomorphism is spatial by Lemma 2.

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