APPLICATION OF SERRIN'S KERNEL PARAMETRIX TO THE UNIQUENESS OF L_1 SOLUTIONS OF ELLIPTIC EQUATIONS IN THE UNIT BALL

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ABSTRACT. In this paper it will be established that L_1 solutions of elliptic partial differential equations, with α -Hölder continuous coefficients, which assume their boundary values mean continuously on the boundary of the N -dimensional unit ball are uniquely determined. An additional application of the kernel will be to establish the Fatou radial limit theorem.

1. Preliminaries. We shall work in N -dimensional Euclidean space R^N , $3 \leq N$, and shall use the following notation: $x = (x_1, \dots, x_N)$ and $B(x, r) =$ the open N -ball centered at x with radius r ; $\tilde{B}(x, r) = B(x, r) \cap B(0, 1)$; $|E|$, the Lebesgue measure of E ; ∂E , the boundary of E ; $\bar{\partial} B(x, r) = \partial B(0, 1) \cap B(x, r)$; $d\omega$, the natural surface area on $\partial B(0, 1)$ and $dS_r = r^{N-1} d\omega$; and sub-scripted A 's, positive constants which depend only on N and the operator L defined below.

For a point $y_0 \in \partial B(0, 1)$, $u(x)$ integrable on $B(0, 1)$, and $f(y)$ a finite valued function on $\partial B(0, 1)$, we set

$$u_f(y_0, \rho) = |\tilde{B}(y_0, \rho)|^{-1} \int_{\tilde{B}(y_0, \rho)} |u(x) - f(y_0)| dx, \quad 0 < \rho \leq 1.$$

If $f(y)$ is 0 a.e. $[d\omega]$, we use the notation $u(y_0, \rho)$. We say that $u(x)$ assumes the boundary limits $f(y)$ mean continuously on $\partial B(0, 1)$ if

- (a) $u_f(y_0, \rho) = o(1)$ as $\rho \rightarrow 0$ a.e. $[d\omega]$, and
- (b) $u_f(y_0, \rho) = O(1)$ as $\rho \rightarrow 0$ for each $y_0 \in \partial B(0, 1)$.¹

Let L denote the elliptic partial differential operator

$$\sum_{i,j}^N a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i^N b_i(x) \frac{\partial}{\partial x_i} + c(x) \quad (c(x) \leq 0)$$

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¹See [1] for a related use of mean continuous limits.

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whose coefficients are α -Hölder continuous, $0 < \alpha \leq 1$, on $\overline{B(0, 1)}$; as such, L is uniformly elliptic, i.e., there is a positive constant λ such that for any ξ and any $\xi \neq 0$

$$\lambda^{-1} \sum_{i=1}^N \xi_i^2 \leq \sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \leq \lambda \sum_{i=1}^N \xi_i^2.$$

The main result of this paper is

Theorem 1. *There is at most one function $u(x)$ satisfying $Lu = 0$ in $B(0, 1)$ which assumes the boundary limits $f(y)$ mean continuously on $\partial B(0, 1)$.*

Theorem 2. *If $Lu = 0$ in $B(0, 1)$ and $\sup_{0 \leq r < 1} \int_{\partial B(0, r)} |u(x)| dS_r(x) < \infty$, then u has radial limits a.e. $[d\omega]$.*

Remarks. In establishing Theorem 1, no appeal is made to the existence theorems for the operator L ; the main tool which we use is the kernel parametrix introduced by J. Serrin [3, p. 301]. The Poisson kernel for harmonic functions shows that condition (b) cannot be weakened to "for all except one $y_0 \in \partial B(0, 1)$ ". See [5] for more general results than Theorem 2.

2. Lemmas. The essential properties of the kernel parametrix which will be needed are as follows (see [3] for proof):

Let $B(x_0, a) \subset B(0, 1)$, $\rho = |x - x_0|$, and $R^2 = \sum_{i,j=1}^N A_{ij}(y)(x_i - y_i)(x_j - y_j)$, where (A_{ij}) is the inverse matrix of (a_{ij}) and $y \in \partial B(x_0, a)$. The kernel parametrix

$$k(x, y) = (a^2 - \rho^2)a^{-1}R^{-N}$$

is used in composition with two functions $f_{\pm}$ of a real variable with $f_{\pm}(0) = 0$, $f'_{\pm} > 0$, such that

- (1) $0 \leq Lf_{+}(k(x, y))$ and $Lf_{-}(k(x, y)) \leq 0$ for y fixed;
- (2) for any point $y_0 \in \partial B(x_0, a)$ and $g(y)$ continuous,

$$\lim_{x \rightarrow y_0} \int_{\partial B(x_0, a)} f_{\pm}(k(x, y))g(y) dS_a(y) = g(y_0);$$

- (3) there are constants A_1 and A_2 such that

$$A_1 k(x, y) \leq f_{\pm}(k(x, y)) \leq A_2 k(x, y).$$

Let $K_{\pm}(x, y)$ denote $f_{\pm}(k(x, y))$.

Lemma 1. *If $Lu = 0$ in $B(x_1, a) \subset B(0, 1)$, then*

$$(4) \quad |u(x_1)| \leq A_3 |B(x_1, a)|^{-1} \int_{B(x_1, a)} |u(y)| dy.$$

Proof. Let $K_-^r(x, y)$ be the kernel for $B(x_1, r)$, $0 < r \leq a$. By (1), (2), and the maximum principle

$$-\int_{\partial B(x_1, r)} K_-^r(x, y) |u(y)| dS_r(y) \leq u(x) \leq \int_{\partial B(x_1, r)} K_-^r(x, y) |u(y)| dS_r(y).$$

Thus

$$\begin{aligned} |u(x_1)| &\leq \int_{\partial B(x_1, r)} K_-^r(x_1, y) |u(y)| dS_r(y) \\ &\leq A_2 \int_{\partial B(x_1, r)} r R^{-N} |u(y)| dS_r(y) \quad (\text{by (3)}) \\ &\leq A_2 \lambda^{N/2} r^{1-N} \int_{\partial B(x_1, r)} |u(y)| dS_r(y). \end{aligned}$$

Integrating clearly yields (4).

Lemma 2. *If μ is a signed measure of bounded variation on $\partial B(0, 1)$, then*

$$(5) \quad \lim_{r \rightarrow 1} \int_{\partial B(0, 1)} K_{\pm}(ry_0, y) d\mu(y) = d\mu(y_0)/d\omega \quad \text{a.e. } [d\omega].$$

Proof. Let μ_a and μ_s denote the absolutely continuous and singular parts of μ ; let $\mu_s = \mu_s^+ - \mu_s^-$ be the Hahn decomposition of μ_s . Let $g(y)$ be the derivative of μ_a and select y_0 so that $d\mu_a(y_0)/d\omega = g(y_0)$, y_0 a Lebesgue point of $g(y)$, and $d\mu_s^{\pm}(y_0)/d\omega = 0$. By (2) we can assume that $g(y_0) = 0$.

For μ_s^+ we have by (3), uniform ellipticity, and $d\mu_s^+(y_0)/d\omega = 0$, that

$$0 \leq \int_{\partial B(0, 1)} K_{\pm}(ry_0, y) d\mu_s^+(y) \leq A \int_{\partial B(0, 1)} PK(ry_0, y) d\mu_s^+(y) = o(1)$$

as $r \rightarrow 1$, where PK is the Poisson kernel for Laplace's equation; see [2, Chapter 2]. The same holds for $d\mu_s^-$.

For μ_a we have again by (3) and uniform ellipticity

$$\left| \int_{\partial B(0,1)} K_{\pm}(ry_0, y) d\mu_a(y) \right| = \left| \int_{\partial B(0,1)} K_{\pm}(ry_0, y) g(y) d\omega(y) \right|$$

$$\leq A \int_{\partial B(0,1)} PK(ry_0, y) |g(y)| d\omega(y) = o(1) \quad \text{as } r \rightarrow 1,$$

since y_0 is a Lebesgue point of $g(y)$, thus yielding (5).

Lemma 3. *If $Lu = 0$ and $|u(x)| \leq M < \infty$ in $\tilde{B}(y_0, a)$, $|y_0| = 1$, and if $u(x)$ assumes the boundary limits $f(y) \equiv 0$ mean continuously on $\tilde{\partial}B(y_0, a)$, then $u(x)$ is continuously 0 on $\tilde{\partial}B(y_0, a)$.*

Proof. The procedure will be to construct a majorant and minorant of $u(x)$ which are continuously 0 on $\tilde{\partial}B(y_0, a)$.

Let $K_{\pm}^r(x, y)$ denote the kernels for $B(0, r)$. Let $C_i(r, \rho)$, $i = 1, 2$, denote the closed spherical caps forming the boundary of $B(0, r) \cap \tilde{B}(y_0, \rho)$, if not empty, with $C_1(r, \rho) \subseteq \partial B(0, r)$. Let $0 \leq \lambda(ry) \leq 1$, $|y| = 1$, be a smoothing function which is 1 on $C_1(r, a/3)$ and 0 on $\partial B(0, r) - C_1(r, a/2)$. Set

$$u_+^r(y) = \max [\lambda(ry)u(ry), 0]; \quad u_-^r(y) = -\min [\lambda(ry)u(ry), 0];$$

form

$$v_{\pm}^r(x) = r^N \int_{\partial B(0,1)} K_{\pm}^r(x, ry) u_{\pm}^r(y) d\omega(y) - r^N \int_{\partial B(0,1)} K_{\mp}^r(x, ry) u_{\mp}^r(y) d\omega(y)$$

for $|x| < r$. Since $|u_{\pm}^r(y)| \leq M$ and $K_{\pm}^r(x, ry) \leq APK(x, ry)$ for all $r < 1$, $|v_{\pm}^r(x)| \leq M_1 = 2AM$ by maximum principle for harmonic functions. Furthermore, by (2), for $x \in C_1(r, a/3)$, $v_{\pm}^r(x) = u(x)$ and $Lv_-^r(x) \leq 0$, $0 \leq Lv_+^r(x)$.

We can select $g(y)$ to be 0 on the closure of $\tilde{\partial}B(y_0, a/4)$ and positive otherwise, in such a way that

$$(6) \quad 2M_1 \leq G(x) = \int_{\partial B(0,1)} K_-(x, y) g(y) d\omega(y),$$

for $x \in B(0, 1) - \tilde{B}(y_0, a/3)$.

Finally, set $w_{\pm}^r(x) = v_{\pm}^r(x) \mp G(x)$. For $w_{\pm}^r(x)$ we have that

- (i) $0 \leq Lw_+^r(x)$ and $Lw_-^r(x) \leq 0$,
- (ii) $w_+^r(x) \leq u(x) \leq w_-^r(x)$ for $x \in B(0, r) \cap \tilde{B}(y_0, a/3)$, and
- (iii) $\lim_{r \rightarrow 1} w_{\pm}^r(x) = \mp G(x)$ for $|x| < 1$.

By construction of $w_{\pm}^r(x)$, (i) is obvious. For $x \in C_1(r, a/3)$, $w_{\pm}^r(x) = u(x) \mp G(x)$, and since $G(x)$ is positive, (ii) holds on $C_1(r, a/3)$. For $x \in C_2(r, a/3)$,

$$u(x) \leq M \leq M_1 \leq G(x) - |v_-^r(x)| \leq w_-^r(x);$$

likewise

$$w_+^r(x) \leq |v_+^r(x)| - G(x) \leq -M_1 \leq -M \leq u(x).$$

Hence by (i) and the maximum principle, (ii) holds. To show (iii), fix x , $|x| < 1$. For $y \in \partial B(0, 1)$, by Lemma 1,

$$|u(ry)| \leq A_3 |B(ry, 1-r)|^{-1} \int_{B(ry, 1-r)} |u(y)| dy \leq 2^N A_3 u(y, 2(1-r)).$$

This together with (a) yields $\lim_{r \rightarrow 1} |u(ry)| = 0$ a.e. $[d\omega]$. Thus $\lim_{r \rightarrow 1} |u_{\pm}^r(y)| = 0$ a.e. $[d\omega]$ and $u_{\pm}^r(y)$ are bounded for $r < 1$. Since for $|x| < 1$, $K_{\pm}^r(x, y)$ are bounded, we have by the dominated convergence theorem that $\lim_{r \rightarrow 1} v_{\pm}^r(x) = 0$ which establishes (iii). By (ii) and (iii), $-G(x) \leq u(x) \leq G(x)$, for $x \in \tilde{B}(y_0, a/3)$ and $G(x)$ is continuously 0 on $\bar{\partial}B(y_0, a/4)$; so $u(x)$ is continuously 0 on this set, and the lemma easily follows.

3. Proof of Theorem 1. Let u_1 and u_2 be two solutions, $Lu_i = 0$, which assume the boundary values $f(y)$ mean continuously on $\partial B(0, 1)$. Set $u = u_1 - u_2$. Clearly u assumes the boundary value 0 mean continuously on $\partial B(0, 1)$. If we can show that u is bounded in $B(0, 1)$, then since

$$v_+^r(x) \leq u(x) \leq v_-^r(x) \quad \text{for } |x| \leq r,$$

where $v_{\pm}^r(x)$ is defined as in Lemma 3 with $u_+^r(y) = \max\{u(ry), 0\}$ and $u_-^r(y) = -\min\{u(ry), 0\}$, by the dominated convergence theorem $\lim_{r \rightarrow 1} v_{\pm}^r(x) = 0$ for each $|x| < 1$, which gives $u \equiv 0$.

Let Z be the set of points z on $\partial B(0, 1)$ such that $u(x)$ is unbounded in every neighborhood $\tilde{B}(z, \rho)$. Z is a closed set. Assume $Z \neq \emptyset$, for otherwise u is bounded in $B(0, 1)$. Set $g_n(y) = u(y, 1/n)$. Since $u \in L_1$, by continuity of the integral, $g_n(y)$ is continuous on $\partial B(0, 1)$ for $n = 1, 2, 3, \dots$; let $g_n(z)$ be its restriction to the set Z . By the Baire category theorem, since $Z \neq \emptyset$, there is a $z_0 \in Z$, a $0 < \rho_0$, and a constant K such that $g_n(z) \leq K$ for $z \in \tilde{\partial}B(z_0, \rho_0) \cap Z$ and $n = 1, 2, 3, \dots$ ²

We intend to show that $u(x)$ is bounded in $\tilde{B}(z_0, \rho_0/4)$ which contradicts $z_0 \in Z$. Select $x_1 \in \tilde{B}(z_0, \rho_0/4)$ and let ρ_1 be the largest value for which $B(x_1, \rho_1) \cap Z = \emptyset$. Clearly since $\rho_1 \leq \rho_0/4$, there is a point $z^* \in \partial B(x_1, \rho_1) \cap \tilde{\partial}B(z_0, \rho_0) \cap Z$. Set $u_0(x) = u(x)$ for $x \in B(x_1, \rho_1) \cap B(0, 1)$ and $u_0(x) \equiv 0$ for $x \in B(x_1, \rho_1) - B(0, 1)$. Since $B(x_1, \rho_1) \cap Z = \emptyset$, by Lemma 3, $u_0(x)$ is continuous in $B(x_1, \rho_1)$. Even though $u_0(x)$ does not satisfy $Lu_0 = 0$ in $B(x_1, \rho_1)$ we have

²See [4] for the origin of this approach.

$$(7) \quad |u_0(x_1)| \leq A_3 |B(x_1, \rho)|^{-1} \int_{B(x_1, \rho)} |u_0(y)| dy, \quad \rho \leq \rho_1;$$

to see this, extend L^α -Hölder continuously to $B(x_1, \rho_1)$; then

$$(8) \quad -\int_{\partial B(x_1, r)} K_-^r(x, y) |u_0(y)| dS_r(y) \leq u_0(x) \leq \int_{\partial B(x_1, r)} K_-^r(x, y) |u_0(y)| dS_r(y)$$

for $x \in \partial \tilde{B}(x_1, r)$ and $r < \rho_1$ since $u_0(x) \equiv 0$ on $\partial \bar{B}(x_1, \rho_1)$. Thus by the maximum principle, (8) holds for $x \in \tilde{B}(x_1, r)$ and in particular for $x = x_1$; so (7) follows from (8) as in Lemma 1.

From (7) and the definition of $u_0(x)$ we have

$$(9) \quad |u(x_1)| \leq A_3 |B(x_1, \rho)|^{-1} \int_{\tilde{B}(x_1, \rho)} |u(y)| dy, \quad \rho \leq \rho_1.$$

Select n so that $1/2n < \rho_1 \leq 1/2(n-1)$. Then by (9)

$$\begin{aligned} |u(x_1)| &\leq 2A_3 |\tilde{B}(x_1, 1/2n)|^{-1} \int_{\tilde{B}(x_1, 1/2n)} |u(y)| dy \\ &\leq 2A_3 |\tilde{B}(x_1, 1/2n)|^{-1} \int_{\tilde{B}(z^*, 1/n-1)} |u(y)| dy \\ &\leq 2^{2N+1} A_3 g_{n-1}(z^*) \leq A_4 K, \end{aligned}$$

which establishes a bound for u in $\tilde{B}(z_0, \rho_0/4)$.

4. **Proof of Theorem 2.** As in the previous proof, we have $v_+^r(x) \leq u(x) \leq v_-^r(x)$ for $|x| < r$. As in [2, pp. 26–28] there are positive measures μ_+ and μ_- giving rise to functions $v_\pm(x)$ such that $v_+(x) \leq u(x) \leq v_-(x)$, and the theorem follows immediately from Lemma 2.

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