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THE F. AND M. RIESZ THEOREM[1]

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ABSTRACT. We weaken the hypothesis of the F. and M. Riesz theorem.

1. If X is a locally compact Hausdorff space, then we will denote by $M_+(X)$ the class of all Radon measures on X. Thus if $\mu \in M_+(X)$ and $E \subset X$, then $\mu(E) \geq 0$. We will denote by $M(X)$ the complex linear span of those μ in $M_+(X)$ for which $\mu(X) < \infty$. If X is a locally compact abelian group, and if $\mu \in M(X)$, then we will denote (as is usual) by $\hat{\mu}$ the Fourier transform of μ. We let $\mathbf{Z}_+ = \{k: k \in \mathbf{Z}, k > 0\}$. We will denote by γ the Lebesgue measure on $\mathbf{R}$. The well-known F. and M. Riesz theorem (in one of its many versions) states that if $\mu \in M(\mathbf{R})$ and if $\hat{\mu} = 0$ on $(0, \infty)$, then $\mu \ll \gamma$. The purpose of this paper is to point out that the condition on $\hat{\mu}$ may be weakened as follows.

2. **Theorem.**[2] *Let a and b in $(0, \infty)$ be linearly independent over $\mathbf{Z}$, and let $S = \{ja + kb: (j, k) \in \mathbf{Z}_+ \times \mathbf{Z}_+\}$. If $\mu \in M(\mathbf{R})$ and if $\hat{\mu} = 0$ on S, then $\mu \ll \gamma$.*

3. With regard to Theorem 2 we remark that if $t > 0$, then $(0, t) \cap S$ is finite.

4. We will now prove Theorem 2. We define $\phi: \mathbf{R} \to \mathbf{T} \times \mathbf{T}$ by $\phi(t) = (e^{iat}, e^{ibt})$ and we let $\lambda = \mu \circ \phi^{-1}$. Thus $\lambda \in M(\mathbf{T} \times \mathbf{T})$. If $f \in C(\mathbf{T} \times \mathbf{T})$, then $\int f\, d\lambda = \int f \circ \phi\, d\mu$; hence if $(j, k) \in \mathbf{Z} \times \mathbf{Z}$, then

$$\hat{\lambda}(j, k) = \int \bar{z}^j \bar{w}^k\, d\lambda(z, w) = \int e^{-i(ja+kb)t}\, d\mu(t) = \hat{\mu}(ja + kb).$$

Thus $\hat{\lambda} = 0$ on $\mathbf{Z}_+ \times \mathbf{Z}_+$; hence by [1, Theorem 6.2.2], if $E \subset \mathbf{R}$ and $\gamma(E) = 0$, then $\lambda(\phi(E)) = 0$. Since $E = \phi^{-1}(\phi(E))$, we have $\mu(E) = 0$ which completes the proof of Theorem 2.

REFERENCE

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[2] The referee reports that Theorem 2 has also been obtained by Brian Cole.