PROPERTY Γ AND INNER AMENABILITY

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ABSTRACT. If the regular group von Neumann algebra of a countable, infinite conjugacy class group satisfies Property Γ , then the group has a nontrivial mean which is invariant under inner automorphisms.

1. **Introduction.** Group algebras provide some of the most important examples of operator algebras. If G is a countable (discrete) group, let $l^2(G)$ denote the Hilbert space of square summable sequences on G . For each $s \in G$, define $L(s): l^2(G) \rightarrow l^2(G)$ to be left translation by s , i.e., given f in $l^2(G)$, $(L(s)f)(t) = f(s^{-1}t)$ for all t in G . The left group von Neumann algebra, $B(G)$, is the smallest weakly closed algebra of operators on $l^2(G)$ that contains $L(G)$. $B(G)$ is a finite von Neumann algebra, and it is a factor if and only if the conjugacy class of each element of G , other than the identity, is infinite (see [1, p. 282]). If the latter is the case, we will say that G is ICC.

A central problem in analytic algebra is to relate properties of $B(G)$ to those of G . The most elegant theorem in this direction is that of Schwartz [6]: G has an invariant mean if and only if $B(G)$ satisfies a condition known as "Property P". A much earlier result, due to Murray and von Neumann [4], is that for only certain groups G does $B(G)$ satisfy another condition, "Property Γ ". These authors used this information to prove the existence of two, and then of three distinct isomorphism classes of type II_1 factors. Subsequently, McDuff [3] and Sakai [5] used refinements of Γ to distinguish uncountably many such factors. Nonetheless, Γ is poorly understood, since the groups G for which $B(G)$ has Property Γ have not been characterized.

In this note, we will prove that if $B(G)$ is a factor with property Γ , then there is a nontrivial mean on G which is invariant under inner automorphisms. We suspect that the converse is true, but we have been unable to prove it. Despite this uncertainty, the result provides a particularly simple explana-

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tion for why certain groups G , such as the free group on two generators, do not have group von Neumann algebras with Property Γ .

2. Group von Neumann algebras. We will let $l^1(G)$ and $l^\infty(G)$ be the summable and the bounded sequences on G , respectively, and we will write $\|\cdot\|_p$ for the usual norm on $l^p(G)$, $p = 1, 2, \infty$. In addition to the operators $L(s)$, $s \in G$, we have isometries $R(s)$ and $A(s)$ of these spaces defined by

$$(R(s)f)(t) = f(ts), \quad (A(s)f)(t) = f(s^{-1}ts),$$

i.e., $A(s) = L(s)R(s)$.

For each $s \in G$, we let d_s be the function which is 1 at s and 0 elsewhere. Such functions form an orthonormal basis for $l^2(G)$, hence $b \rightarrow bd_e$ is a linear isomorphism of $B(G)$ onto a dense subspace of $l^2(G)$. We will often use this map to identify operators in $B(G)$ with l^2 -functions on G . In particular, $\tau(b) = b(e) = bd_e \cdot d_e$ defines a faithful trace on $B(G)$.

Since the operators $R(s)$, $L(t)$ commute for all $s, t \in G$, the $R(s)$ commute with all elements in $B(G)$, and we have for b in $B(G)$:

$$L(s)bL(s)^{-1}d_e = L(s)bd_{s^{-1}} = L(s)bR(s)d_e = A(s)bd_e,$$

or regarding the following as an equality of l^2 -functions,

$$(2.1) \quad L(s)bL(s)^{-1} = A(s)(b).$$

Although not all l^2 -functions correspond to elements of $B(G)$ (consider the additive group of integers), the situation is quite different for f in $l^1(G)$. For any scalars c_s , we have $\|\sum c_s L(s)\| \leq \sum |c_s|$; hence the sum $L(f) = \sum f(s)L(s)$ converges uniformly to an element in $B(G)$, and the corresponding l^2 -function is just f . Similarly, $A(f) = \sum f(s)A(s)$ converges uniformly to an operator on $l^2(G)$.

A mean m on G is a positive linear functional on $l^\infty(G)$ (and thus an element of the Banach space dual $l^\infty(G)^*$) such that $m(1) = 1$. If D is a subset of G , the characteristic function χ_D is in $l^\infty(G)$, and we write $m(D) = m(\chi_D)$. A mean m on G is said to be invariant if for all $s \in G$, and $f \in l^\infty(G)$,

$$m(L(s)f) = m(R(s)f) = m(f),$$

and if such a mean exists, G is said to be amenable. m is inner invariant if for all $s \in G$, $f \in l^\infty(G)$, $m(A(s)f) = m(f)$. Regarding $l^1(G)$ as a subspace of $l^\infty(G)^*$, it is evident that d_e is such a mean. We say that an inner invariant mean is nontrivial if $m(\{e\}) \neq 1$, or equivalently, $m \neq d_e$, and if such a mean exists, we say that G is inner amenable.

$$m' = \frac{m - m(\{e\})d_e}{1 - m(\{e\})}$$

will then be inner invariant with $m'(\{e\}) = 0$.

Any invariant mean on G is a nontrivial inner invariant mean, hence the amenable groups are inner amenable. The free group on two generators, F_2 , is not inner amenable. To see this, note that there exists a subset D and elements s_1, s_2, s_3 of G such that $F_2 = D \cup s_1 D s_1^{-1} \cup \{e\}$, but the sets D , $s_2 D s_2^{-1}$, and $s_3 D s_3^{-1}$ are disjoint (see [1, p. 285]—this is the heart of the argument for why $B(F_2)$ does not satisfy Γ). If F_2 is inner amenable, let m be an inner invariant mean with $m(\{e\}) = 0$. Then

$$2m(D) + m(\{e\}) \geq m(F_2) = 1 \geq 3m(D),$$

from which we conclude that $0 = 2m(D) \geq m(F_2) = 1$, a contradiction. For an example of an ICC group which is inner amenable but not amenable, consider $G \times F_2$, where G is the group of finite permutations of the integers. G is amenable (see [6]) and thus has an invariant mean m . $G \times F_2$ is not amenable, but $f \rightarrow m(f(\cdot, e))$ provides a nontrivial inner invariant mean for $f \in l^\infty(G \times F_2)$.

Let B be a finite factor with normalized trace τ . We define a pre-Hilbert norm $\|\cdot\|_2$ on B by $\|b\|_2 = \tau(b^*b)^{1/2}$. If $B = B(G)$, and $\tau(b) = b(e)$, this coincides with the $l^2(G)$ norm. B is said to have Property Γ if for each $b_1, \dots, b_n \in B$, and $\epsilon > 0$, there exists a unitary $u \in B$ such that $\tau(u) = 0$, and $\|ub_j u^* - b_j\|_2 < \epsilon$, $j = 1, \dots, n$.

Theorem. Suppose that G is an ICC group. Then consider the following conditions:

- (1) $B(G)$ has Property Γ .
- (2) There exists a sequence $g_n \in l^2(G)$ such that $\|g_n\|_2 = 1$, $g_n(e) = 0$ and $\|A(s)g_n - g_n\|_2 \rightarrow 0$ for each $s \in G$.
- (3) There exists a sequence $h_n \in l^1(G)$ such that $h_n \geq 0$, $\|h_n\|_1 = 1$, $h_n(e) = 0$, and $\|A(s)h_n - h_n\|_1 \rightarrow 0$.
- (4) G is inner amenable.

We have $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$.

Proof. $(1) \Rightarrow (2)$. Let $G = \{s_1, s_2, \dots\}$. Given n , choose a unitary $g_n \in B(G)$ such that $\tau(g_n) = 0$, and

$$\|g_n L(s_j) g_n^* - L(s_j)\|_2 < 1/n, \quad j = 1, \dots, n.$$

Since multiplication by a unitary is isometric, we have from (2.1),

$$\begin{aligned} \|g_n - A(s_j)g_n\|_2 &= \|g_n - L(s_j)g_n L(s_j)^{-1}\|_2 \\ &= \|g_n L(s_j) g_n^* - L(s_j)\|_2 < 1/n, \quad j = 1, \dots, n. \end{aligned}$$

The l^2 -norm of a unitary must be one, and $g_n(e) = \tau(g_n) = 0$, hence g_n is the desired sequence.

(2) $\Rightarrow$ (3). Letting $h_n = |g_n|^2$, the first three properties are evident, and

$$\begin{aligned} \|A(s)h_n - h_n\|_1 &\leq \|A(s)|g_n| - |g_n|\|_2 \|A(s)|g_n| + |g_n|\|_2 \\ &\leq 2\|A(s)g_n - g_n\|_2 \rightarrow 0. \end{aligned}$$

(3) $\Rightarrow$ (2). If we let $g_n = h_n^{1/2}$, the first two properties are immediate and, as in [2, p. 46],

$$\|A(s)g_n - g_n\|_2^2 \leq \|A(s)h_n - h_n\|_1 \rightarrow 0.$$

(3) $\Leftrightarrow$ (4). This is essentially Day's "convergence to invariance" result (see [2, p. 33]). The desired mean is any weak* limit point of the sequence h_n . Conversely, the weak* density of the means in $l^1(G)$ among those in $l^\infty(G)^*$ provide a net of weakly* approximating h_n to a given inner invariant mean. The desired differences converge weakly to 0, and since the weak and norm closure of a convex set coincide, convex combinations will converge in norm. Since G is countable, one may extract a sequence from the net.

The only new wrinkle, is that if we assume $h_n(e) = 0$, then $m(\{e\}) = \lim h_n \cdot e = 0$, and conversely, if we assume $m(\{e\}) = 0$, we may also assume $h_n(e) = 0$.

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