AN EXCHANGE THEOREM FOR INDEPENDENCE STRUCTURES

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ABSTRACT. I give a simple proof of the following result, which extends a result of C. Greene [1]. Let X and Y be sets in an independence structure $\mathfrak{E}$ and let $X = X_1 \cup X_2$ be a partition of X . Then there exists a partition $Y = Y_1 \cup Y_2$ of Y such that for $i = 1, 2$, $X_i \cap Y_i = \emptyset$ and $X_i \cup Y_i$ is in $\mathfrak{E}$.

I give a simple proof of a result for independence structures (not necessarily finite) which extends a known result for matroids. The matroid result is stated and proved by C. Greene [1], and given a short proof by D. R. Woodall [3]. My proof is based on a simplification of Woodall's proof.

Theorem. Let $\mathfrak{E}, \rho$ be an independence structure, let X and Y be sets in $\mathfrak{E}$, and let $X = X_1 \cup X_2$ be a partition of X . Then there exists a partition $Y = Y_1 \cup Y_2$ of Y such that for $i = 1, 2$ $X_i \cap Y_i = \emptyset$ and $X_i \cup Y_i$ is in $\mathfrak{E}$.

Note that if $\mathfrak{E}$ has finite rank and X and Y are bases of $\mathfrak{E}$, then it is immediate that $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are bases of $\mathfrak{E}$, since

$$|X_1 \cup Y_1| + |X_2 \cup Y_2| = |X_1| + |Y_1| + |X_2| + |Y_2| = |X| + |Y|.$$

This is essentially the result proved in [1] and [3]. To prove the Theorem I use the following well-known lemma.

Lemma. Let $\mathfrak{E}, \rho$ be an independence structure on a set S , let $X \in \mathfrak{E}$ and let $\mathfrak{E}'$ be the contraction of $\mathfrak{E}$ from X , that is $\mathfrak{E}' = \{A \subseteq S \setminus X: A \cup X \in \mathfrak{E}\}$. Then $\mathfrak{E}'$ is an independence structure on $S \setminus X$, with rank function ρ' , say, and for each finite subset A of $S \setminus X$ there exists a finite subset B of S such that $\rho'A = \rho(A \cup (X \cap B)) - |X \cap B|$.

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Proof of Lemma. Easy, using the finite character of $\mathfrak{E}$.

Proof of Theorem. For $i = 1, 2$, let $\mathfrak{E}_i$, ρ_i be the contraction of $\mathfrak{E}$ from X_i . We wish to show that Y is in the sum $\mathfrak{E}_1 \vee \mathfrak{E}_2$. Now this happens [2] if and only if for each finite subset A of Y , $\rho_1 A + \rho_2 A \geq |A|$. But for each finite subset A of Y , and some finite set B , using the submodularity of ρ ,

$$\begin{aligned} \rho_1 A + \rho_2 A &= \rho(A \cup (X_1 \cap B)) - |X_1 \cap B| + \rho(A \cup (X_2 \cap B)) - |X_2 \cap B| \\ &\geq \rho(X \cap B) + \rho A - |X \cap B| = |A|. \end{aligned}$$

REFERENCES

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