SOME PROPERTIES OF CERTAIN RELATIVELY FREE GROUPS

T. C. HURLEY

ABSTRACT. It is shown that the groups $F/[F_n, F_m]$, where F is a free group and m, n are positive integers such that $m < n \leq 2m$, are residually "torsion-free and nilpotent", and the structure of their lower central factors is computed.

Let m, n be positive integers such that $m < n \leq 2m$. The groups $F/[F_n, F_m]$, where F is a free group, are studied. It is shown that $[F_{n-1}, F_m]/[F_n, F_m]$ is a free abelian group and a basis is given. The structure of the lower central factors of $F/[F_n, F_m]$ is computed, and it is shown that they are free abelian groups. Then, by showing it is residually nilpotent, it is deduced that $F/[F_n, F_m]$ is residually "torsion-free and nilpotent" and, hence, that it is residually a finite p -group for all primes p .

If a and b are elements of a group G , then the commutator $[a, b] = a^{-1}b^{-1}ab$. Higher commutators are left-normed. If H and K are subsets of G , then $[H, K]$ is the subgroup of G generated by all $[h, k]$ with h in H and k in K . The lower central series of G is defined by $G_1 = G$, $G_{n+1} = [G_n, G]$. The n th lower central factor of G is G_n/G_{n+1} . Throughout, F is a free group of at most countable rank, freely generated by $X = \{x_1, x_2, \dots\}$. The reader is referred to Marshall Hall [4, pp. 165–175] for the definition and properties of basic commutators. Note that the basic commutators are ordered according to weight, and a definite ordering is fixed on the basic commutators of the same weight. If a and b are basic commutators then $a \geq b$ ($a > b$) means that a is greater than or equal to b (respectively, greater than but not equal to b) in the ordering of the basic commutators. Assume throughout that m and n are positive integers with $m \leq n \leq 2m$; for Lemmas 2, 3, 4, 5 and Theorem 6 it is further assumed that $m < n$.

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The result

$$[G_i, G_j] \subseteq G_{i+j} \quad [4, \text{Corollary 10.3.5}]$$

will be used frequently without further reference. A group G is said to be "residually P ", where P is some property of groups, if, for every $g \neq 1$ in G , there exists a normal subgroup N_g of G such that G/N_g has the property P (see [5, pp. 27-32]).

Lemma 1. *If $x \in G_k$, $y \in G_p$ and $z \in G_l$ then*

$$\begin{aligned} [x, y, z] &\equiv [y, z, x]^{-1} [x, z, y] \text{ modulo } [[G_k, G_p], [G_l, G_p]] \\ &\quad \times [[G_k, G_l], [G_l, G_p]] \times [[G_k, G_l], [G_k, G_p]]. \end{aligned}$$

Proof. By the commutator identities in [4, p. 150],

$$\begin{aligned} [x, yz] &= [x, z][x, y][x, y, z] = [x, zy[y, z]] \\ &= [x, [y, z]][x, zy][x, zy, [y, z]] \\ &= [x, [y, z]][x, y][x, z][x, zy][[x, y][x, z][x, y, z], [y, z]] \\ &\equiv [x, [y, z]][x, y][x, z][x, z, y] \\ &\quad \text{modulo } [[G_k, G_p], [G_l, G_p]] \times [[G_k, G_l], [G_l, G_p]] \\ &\equiv [x, z][x, y][y, z, x]^{-1} [x, z, y] \\ &\quad \text{modulo } [[G_k, G_l], [G_k, G_p]] \times [[G_k, G_p], [G_l, G_p]] \times [[G_k, G_l], [G_l, G_p]] \end{aligned}$$

The result now follows. $\square$

The following lemma is easily deduced from the commutator identities.

Lemma 2. *If $a, b \in G_{n-1}$, $c, d \in G_m$, then*

$$[ab, cd] \equiv [a, c][a, d][b, c][b, d] \text{ modulo } [G_n, G_m].$$

If, further, $b \in G_n$, then

$$[ab, cd] \equiv [a, c][a, d] \text{ modulo } [G_n, G_m],$$

and if $d \in G_n$ then

$$[ab, cd] \equiv [a, c][b, c] \text{ modulo } [G_n, G_m].$$

Also

$$[a^{-1}, c] \equiv [a, c]^{-1} \equiv [a, c^{-1}] \text{ modulo } [G_n, G_m].$$

Lemma 3. *Let a be a basic commutator of weight t such that $a = [a_1, c]$, where a_1 and c are basic commutators of weights $t - q$ and q ,*

respectively, and let b be a basic commutator of weight p such that $a > b$ and such that $p \leq t < 3p$. Then either $[a, b]$ is a basic commutator or else

$$[a, b] \equiv [c, b, a_1]^{-1} [a_1, b, c] \text{ modulo } [F_{t+1}, F_p],$$

where $[c, b, a_1]$ and $[a_1, b, c]$ are basic commutators.

Proof. $[a, b]$ is a basic commutator if $c \leq b$. If $c > b$, then weight $c \geq p$, and thus $2p > \text{weight } a_1 \geq \text{weight } c$; this implies that $[c, b, a_1]$ and $[a_1, b, c]$ are basic commutators, and by Lemma 1,

$$\begin{aligned} [a_1, c, b] &\equiv [c, b, a_1]^{-1} [a_1, b, c] \text{ modulo } [[F_{t-q}, F_q], [F_p, F_q]] \\ &\quad \times [[F_{t-q}, F_p], [F_p, F_q]] \times [[F_{t-q}, F_p], [F_{t-q}, F_q]]. \end{aligned}$$

By the *three subgroup lemma* [4, Theorem 10.3.5, p. 155], the group on the right side of the above equation is contained in $[F_{t+1}, F_p]$. (Note that $2p > t - q \geq q \geq p$ and, hence, $F_{t-q} \subseteq F_q \subseteq F_p$.) $\square$

Set

$$F(n + m + k) = \prod_{i+j=k} [F_{n+i}, F_{m+j}]$$

and

$S = \{[x, y]/x \text{ a basic commutator of weight } n - 1 \text{ and}$

$y \text{ a basic commutator of weight } t, m \leq t \leq n - 1\}$,

and for each $k, k = -1, 0, \dots, n - m - 2$, let

$S_k = \{[x, y]/x \text{ a basic commutator of weight } n - 1 \text{ and}$

$y \text{ a basic commutator of weight } m + k + 1\}$.

Then $S = \bigcup_{i=-1}^{n-m-2} S_i$, where the union is disjoint.

Also define, for $k = 0, 1, \dots, n - m - 2$,

$T_k = \{[x, y]/x \text{ a basic commutator of weight } n + i, y \text{ a basic}$

commutator of weight $m + j$, with $i + j = k\}$.

Lemma 4. $[F_{n-1}, F_m]/[F_n, F_m]$ is generated by S .

Proof. $[F_{n-1}, F_m]$ is generated by $[a, b]$, $a \in F_{n-1}$, $b \in F_m$. Now a is congruent to a product of basic commutators of weight $n - 1$ modulo F_n , and b is congruent to a product of basic commutators of weights $m, m + 1, \dots, n - 1$ modulo F_n . Applying Lemma 2 now gives the required result.

Lemma 5. $F(n + m + k)/F(n + m + k + 1)$ is generated by the elements of T_k that are basic ($0 \leq k \leq n - m - 2$).

Proof. It is clear that $F(n+m+k)/F(n+m+k+1)$ is generated by T_k . If $[a, b] \in T_k$ and $[a, b]$ is not a basic commutator, then $a = [a_1, c]$, where $c > b$. Let weight $a = n+i$, weight $b = m+j$, and weight $c = q$ ($i+j=k$).

Then by Lemma 3,

$$[a_1, c, b] \equiv [c, b, a_1]^{-1}[a_1, b, c] \text{ modulo } [F_{n+i+1}, F_{m+j}],$$

where $[c, b, a_1]$ and $[a_1, b, c]$ are basic commutators.

Now weight $a_1 \geq \text{weight } c \geq \text{weight } b \geq m$ and $2m \geq n$. Hence $[c, b, a_1]$ and $[a_1, b, c]$ are contained in T_k ; this completes the proof.

Corollary. $[F_n, F_m]$ is generated, modulo $F(n+m+k+1)$, by the elements of $T_0 \cup T_1 \cup \dots \cup T_k$ that are basic ($0 \leq k \leq n-m-2$).

We are now in a position to prove

Theorem 6. $[F_{n-1}, F_m]/[F_n, F_m]$ is freely generated as an abelian group by S .

Proof. This follows directly from the basis theorem [4, Theorem 11.2.4, p. 175], the Corollary to Lemma 5, the fact that the elements of S are basic commutators, that $S_k \cap T_k = \emptyset$ (null set), and that $F(n+m+k) \subseteq F_{n+m+k}$.

The l th lower central factor of $F/[F_n, F_m]$ is $F_l[F_n, F_m]/F_{l+1}[F_n, F_m]$. It is clear that this group (call it $G(n, m, l)$) is generated by

$$R(n, m, l) = \{x/x \text{ a basic commutator of weight } l \text{ and if } x = [y, z]$$

where y and z are basic commutators with weight $z \geq m$,

then weight $y < n\}$.

It is known (Šmel'kin [3]) that $G(m, m, l)$ is free abelian with basis $R(m, m, l)$ and we prove here

Theorem 7. $G(n, m, l)$ is free abelian with basis $R(n, m, l)$.

Proof. We proceed by induction on $n-m$. The case $n-m=0$ follows by the above-mentioned result of Šmel'kin. Assume $G(n-1, m, l)$ is generated freely by $R(n-1, m, l)$. By taking the natural map $\phi: G(n, m, l) \rightarrow G(n-1, m, l)$, we see that it suffices to prove that the following set Q is independent modulo $F_{l+1}[F_n, F_m]$:

$$Q = \{x/x \text{ a basic commutator of weight } l \text{ where } x = [y, z],$$

y a basic commutator of weight $n-1$ and

z a basic commutator of weight $l-(n-1) \geq m\}$.

This follows at once from the Corollary to Lemma 5—note that $Q = \emptyset$ if $l \geq 2n - 1$.

Theorem 8. $F/[F_n, F_m]$ is residually nilpotent.

Proof. The proof is by induction on $n - m$. When $n - m = 0$ the result follows from Gruenberg [1]. Suppose $n - m > 0$ and let $g \in F$, $g \notin [F_n, F_m]$. Then we need to find N_g with $[F_n, F_m] \subseteq N_g \triangleleft F$ (where $\triangleleft$ means "normal in") such that F/N_g is nilpotent and $g \notin N_g$. If $g \notin [F_{n-1}, F_m]$, then by the inductive hypothesis there exists N_g with $[F_{n-1}, F_m] \subseteq N_g \triangleleft F$ such that F/N_g is nilpotent and $g \notin N_g$, and this N_g will suffice in this situation.

If $g \in [F_{n-1}, F_m]$ ($g \notin [F_n, F_m]$), then g is a nontrivial product of elements of S (which are basic) modulo $[F_n, F_m]$. By corollary to Lemma 5, $[F_n, F_m]$ is generated modulo F_{2n-1} by the elements of $T_0 \cup T_1 \cup \dots \cup T_{n-m-2}$ that are basic, and since $S \cap (T_0 \cup T_1 \cup \dots \cup T_{n-m-2}) = \emptyset$, it follows that $g \notin F_{2n-1}[F_n, F_m]$. Thus $[F_n, F_m] \subseteq F_{2n-1}[F_n, F_m] \triangleleft F$, $F/F_{2n-1}[F_n, F_m]$ is nilpotent and $g \notin F_{2n-1}[F_n, F_m]$. Hence $N_g = F_{2n-1}[F_n, F_m]$ suffices in this situation.

Theorem 9. $F/[F_n, F_m]$ is residually "torsion-free and nilpotent".

Proof. This follows at once from Theorems 7 and 8.

Corollary. $F/[F_n, F_m]$ is residually a finite p -group for all primes p .

Proof. This need only be proved when F is finitely generated. By Gruenberg [1] a finitely-generated torsion-free nilpotent group is residually a finite p -group for all primes p and a residually "residually a finite p -group" is residually a finite p -group.

Since $F/[F_n, F_m]$ has torsion-free lower central factors (Theorem 7), it follows from a result of P. M. Cohn [2] that the so-called dimension subgroups and the lower central series of $F/[F_n, F_m]$ coincide.

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DEPARTMENT OF PURE MATHEMATICS, THE UNIVERSITY, SHEFFIELD S3 7RH,
ENGLAND

Current address: Department of Mathematics, University College, Belfield,
Dublin 4, Ireland