INEQUALITIES IN DIMENSION THEORY FOR POSETS

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ABSTRACT. The dimension of a poset (X, P) , denoted $\dim(X, P)$, is the minimum number of linear extensions of P whose intersection is P . It follows from Dilworth's decomposition theorem that $\dim(X, P) \leq \text{width}(X, P)$. Hiraguchi showed that $\dim(X, P) \leq |X|/2$. In this paper, A denotes an antichain of (X, P) and E the set of maximal elements. We then prove that $\dim(X, P) \leq |X - A|$; $\dim(X, P) \leq 1 + \text{width}(X - E)$; and $\dim(X, P) \leq 1 + 2 \text{width}(X - A)$. We also construct examples to show that these inequalities are sharp.

1. **Introduction.** Dushnik and Miller [4] defined the dimension of a poset, denoted $\dim(X, P)$ or $\dim X$, to be the minimum number of linear extensions of P whose intersection is P . Equivalently, Ore [7] defined $\dim(X)$ to be the smallest integer k such that (X, P) is isomorphic to a subposet of R^k . We refer the reader to [1], [2], and [8] for other definitions and preliminaries. In this paper we establish inequalities involving dimension, width, height, and cardinality. A number of such inequalities are known and we begin by stating a sampling of them.

Theorem. For any posets X, Y , any chain $C \subseteq X$, and any point $x \in X$, the following inequalities hold.

- (1) $\dim(X - x) \leq \dim X \leq 1 + \dim(X - x)$ [5], [1],
- (2) $\dim X \leq 2 + \dim(X - C)$ [5],
- (3) $\dim X \leq \text{width } X$ [5],
- (4) $\dim X \leq |X|/2$ (Hiraguchi's theorem [5], [1]),
- (5) $\dim(X \times Y) \leq \dim X + \dim Y$.

A poset has dimension one iff it is a chain. If a poset consists of an antichain of at least two points, then its dimension is two. Throughout the remainder of this paper we will assume that X is a poset which is neither a chain nor an antichain. We will use the symbols A and E to denote an arbitrary antichain in X and the set of maximal elements respectively. If

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$|X - A| = 1$, but X is not a chain, then it is trivial to show that $\dim X = 2$. Therefore we will assume that for any antichain $A \subseteq X$, $|X - A| \geq 2$. Furthermore we do not distinguish between a poset and its dual.

2. **Some new inequalities.** In this section we establish some new inequalities for the dimension of a poset.

Lemma 1. *Suppose x and y are incomparable points in a poset X , but for every $z \in X - \{x, y\}$, $z > x$ iff $z > y$ and $z < x$ iff $z < y$. Then $\dim(X - x) = \dim X$ unless $X - x$ is a chain.*

Proof. If $X - x$ is not a chain then $\dim X - x \geq 2$; let $L_1, L_2, \dots, L_t$ be linear extensions of $P|_{X - x} = P'$ whose intersection is P' . In $L_1, L_2, \dots, L_{t-1}$ insert y immediately over x , and in L_t insert y immediately under x . The resulting linear extensions of P intersect to give P , and thus $\dim X \leq \dim X - x$. We note that if $X - x$ is a chain, then $\dim X - x = \dim X - y = 1$, but $\dim X = 2$.

A trivial modification of this argument also proves the following statement.

Lemma 2. *Suppose $x > y$ in P but for every $z \in X - \{x, y\}$, $z > x$ iff $z > y$ and $z < x$ iff $z < y$. Then $\dim X = \dim X - x = \dim X - y$.*

Lemma 3. *If $|X - A| = 2$, then $\dim X = 2$.*

Proof. We may assume without loss of generality that X cannot be reduced by either of the preceding lemmas to a poset with the same dimension as X by having fewer number of points. Then it is easy to see that X is isomorphic to a subposet of one of the following posets.

Figure 1

But the coordinatizations given in Figure 1 show that each of these has dimension 2.

Since the removal of a point cannot decrease the dimension more than one, we have proved the following result.

Theorem 2. *If $|X - A| \geq 2$, then $\dim X \leq |X - A|$.*

Combining this result with the easily obtained bound $\dim X \leq \text{width}(X)$, we have established Hiraguchi's theorem¹ that $\dim X \leq |X|/2$ when $|X| \geq 4$.

We also note that the standard examples of maximal dimensional posets, denoted S_n^0 [2], [8], show that the bounds $\dim X \leq \text{width}(X)$, $\dim X \leq |X - A|$ and $\dim X \leq |X|/2$ are best possible.

Theorem 3. $\dim X \leq \text{width}(X - E) + 1$.

Proof. Let $t = \text{width}(X - E)$; then by Dilworth's theorem [3], there is a partition $X - E = C_1 \cup C_2 \cup \dots \cup C_t$, where each C_i is a chain. For each i , let L_i be a linear extension of P which is a lower extension [1] with respect to C_i . Form a linear extension L_{t+1} of X by placing all maximal elements on top of some linear extension M of $X - E$ and then ordering the maximal elements in L_{t+1} in the reverse order imposed on them by L_t . It is easy to see that $L_1 \cap L_2 \cap \dots \cap L_{t+1} = P$, and the proof of our theorem is complete.

For $w = 1$ and $w = 2$, the following examples show that the bound is best possible.

Figure 2

For $n \geq 3$, we construct a poset Y_n as follows. Y_n has $3n + 2$ points $\{a_1, a_2, \dots, a_n, a_{n+1}\} \cup \{y_1, y_2, \dots, y_n\} \cup \{x_1, x_2, \dots, x_n\} \cup \{p\}$. The points $\{a_i \mid i \leq n\}, \{y_i \mid i \leq n\}$ form a copy of S_n^0 . Each y_i covers x_i ; p covers $a_1, a_2, \dots, a_n$ but $p \nmid a_{n+1}$; and a_{n+1} covers all x 's. We illustrate this construction with the Hasse diagram for Y_3 .

¹ K. P. Bogart first suggested that an elementary proof of Hiraguchi's theorem might be produced by considering the complement of the largest antichain. R. Kimble has independently discovered this result; his proof will appear in his thesis [6].

Figure 3

It is clear that if $E = \{p, a_{n+1}\}$, then $w(Y_n - E) = n$. We now show that $\dim Y_n = n + 1$.

Suppose $\dim Y_n \leq n$; let $L_1, L_2, \dots, L_n$ be linear extensions of Y_n whose intersection is the partial ordering on Y_n . We may assume that the L 's have been numbered so that x_i is over a_i in L_i . Now a_{n+1} is over all x 's; since $y_i \parallel a_{n+1}$ but $y_i < a_j$ for all $j \neq i$, $j \leq n$, y_i is under a_{n+1} in all lists except possibly L_i . Hence we must have y_i over a_{n+1} in L_i . Since $p > y_i$ for all i , this implies p is over a_{n+1} in every L_i . The contradiction shows that $\dim Y_n = n + 1$.

We note that it is straightforward to show that each Y_n is irreducible; i.e., the removal of any point from Y_n lowers the dimension to n . We refer the reader to [9] for details.

Theorem 4. $\dim X \leq 2 \text{ width}(X - A) + 1$.

Proof. Suppose $t = \text{width}(X - A)$ and let $X - A = C_1 \cup C_2 \cup \dots \cup C_t$ be a decomposition into chains. For each i , let L_{2i-1} and L_{2i} be upper and lower extensions, respectively, of C_i . Then let M be an ordering of A which is the reverse of ordering imposed on A by L_{2t} ; then let L_{2t+1} be any linear extension of P whose restriction to A is M . Clearly $L_1 \cap L_2 \cap \dots \cap L_{2t+1} = P$ and the proof of our theorem is complete.

To show that the inequality of Theorem 4 is best possible, we construct for each $n \geq 1, b \geq 1$ a poset $X(n, b)$ as follows. $X(n, b)$ contains a maximal antichain A , and $X(n, b) - A = X_U \cup X_L$ is the natural decomposition into upper and lower halves. X_U and X_L each consist of n incomparable chains with each chain containing b points. Every point in X_U is greater

than every point in X_L . For each ordered pair (S, T) where S is an order ideal of $\hat{X}_U$ and T is an order ideal of X_L , there is a point in A which is less than all points in S and greater than all points in T . We illustrate this definition with the Hasse diagrams for $X(1, 2)$ and $X(2, 1)$.

Figure 4

We note that the width of $X(n, b) - A$ is n . However, it can be shown [10] that for sufficiently large b , $\dim X(n, b) = 2n + 1$.

3. **Some open problems.** Although we have outlined in this paper an elementary proof of Hiraguchi's theorem: $\dim X \leq |X/2|$, it is not known whether or not every poset contains a pair of points whose removal lowers the dimension at most 1.

A second problem involves cartesian products. Although $\dim(X \times Y) \leq \dim X + \dim Y$, it is easy to construct posets X, Y for which $\dim(X \times Y) < \dim X + \dim Y$. (In fact $\dim(S_n^0 \times S_n^0) \leq 2n - 2$.) The question involves the accuracy of the lower bound $\dim(X \times Y) \geq \max\{\dim X, \dim Y\}$.

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