INDUCED REPRESENTATIONS OF DISCRETE GROUPS

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ABSTRACT. We determine necessary and sufficient conditions for a unitary representation of a discrete group induced from a finite-dimensional representation to be irreducible, and also briefly examine the question of when these induced representations are Type II.

Induced representations play an important role in the theory of representations of locally compact groups, and it is of some interest to know when an induced representation is irreducible. In the case where the representation π on the separable locally compact group G is induced from an irreducible representation σ on a normal subgroup H , a simple criterion is known. Let σ^x ($x \in G$) be the representation defined by the action of x on H : $\sigma^x(h) = \sigma(xhx^{-1})$. Then π is irreducible $\Leftrightarrow \sigma^x \cong \sigma$ only if $x \in H$. (See [2] for a proof; $\Rightarrow$ is elementary, but $\Leftarrow$ requires direct integral theory.) In general, of course, anything so simple is too much to hope for. However, in the case where G is discrete, one can hope that at least the question will have some sort of reasonable answer. In this case, one could ask as well for further information about π . In this paper, we find an irreducibility criterion for π in the case where $\dim \sigma$ is finite, and say a bit (but only a bit) about other properties π may have.

Given the subgroup H of the discrete group G , we define the quasnormalizer Q_H of H to be $\{x \in G: x^{-1}Hx \cap H \text{ has finite index in } H \text{ and in } x^{-1}Hx\}$. For convenience, we denote $H \cap x^{-1}Hx$ by H_x . (Then $x \in Q_H \Leftrightarrow H_x$ and $H_{x^{-1}}$ both have finite index in H .)

Lemma 1. H_x has finite index in $H \Leftrightarrow xH$ is contained in finitely many right cosets Hy of $H \Leftrightarrow HxH$ is a finite union of right cosets of H .

Proof. The second equivalence is straightforward, so we work on the first.

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$\Rightarrow$: if $\exists$ elements $x_1, \dots, x_n$ with $H = \bigcup_{j=1}^n H_x x_j$, then let $H_x = x^{-1}Kx$ ($K \subseteq H$), and

$$xH \subseteq \bigcup_{j=1}^n Kxx_j \subseteq \bigcup_{j=1}^n Hxx_j.$$

$\Leftarrow$: It suffices to show that xh_1 and xh_2 are in the same right H -coset ($h_j \in H$) $\Leftrightarrow h_1, h_2$ are in the same right H_x coset. If $xh_1, xh_2 \in Hy$, then $xh_1h_2^{-1}x^{-1} \in H$, or $h_1h_2^{-1} \in x^{-1}Hx$; thus $h_1h_2^{-1} \in H_x$. Conversely, if $h_1, h_2 \in H_x z$, then let $h_j = x^{-1}h'_jxz$ ($h'_j \in H$); $xh_j \in h'_jxz = Hxz$, or xh_1 and xh_2 are in the same right coset.

We might note that $[H: H_{x^{-1}}]$ can be infinite even if $[H: H_x] < \infty$. For instance, let G be the free group on two generators x, y , and H is generated by the elements $x^j y x^{-j}$ ($j = 1, 2, \dots$), then $x^{-1}Hx \supset H$ and $[x^{-1}Hx: H] = \infty$. Hence $H_x = H$, but $[x^{-1}Hx: H_x] = \infty$.

Lemma 2. *If K is a subgroup of H of finite index, then $Q_H = Q_K$.*

Proof. If $x \in Q_K$, then K_x has finite index in K and hence in H , but $K_x \subseteq H_x$. The same argument works with $H_{x^{-1}}$. Hence $Q_K \subseteq Q_H$.

Conversely, $x \in Q_H \Rightarrow H_x$ has finite index in H ; but then $x^{-1}Hx \cap H \cap K = x^{-1}Hx \cap K$ has finite index in $H \cap K = K$, and similarly $x^{-1}Kx \cap K = K_x$ has finite index in $x^{-1}Hx \cap K$. Combining these, K_x has finite index in K ; again, $K_{x^{-1}}$ goes the same way.

Theorem 1. Q_H is a group containing H .

Proof. Since $x \in Q_H \Leftrightarrow H_x, H_{x^{-1}}$ have finite index in H , Q_H is closed under inverses. That $H \subseteq Q_H$ is clear.

If $x, y \in Q_H$, then, for instance, H_x is of finite index in H ; from Lemma 2, $y^{-1}(x^{-1}Hx \cap H)y^{-1} \cap H_x$ is of finite index in H_x . This shows that H_{xy} is of finite index in H . The other half is similar.

Now let σ be a finite-dimensional irreducible representation of H on a complex inner product space $\mathcal{H}_0$, and let π be the corresponding induced representation of G (on $\mathcal{H}$, say). We denote the algebra of intertwining operators of π with itself by $\text{Hom}(\pi, \pi)$. If $A \in \text{Hom}(\pi, \pi)$, then we can describe A as follows: let $\delta_e^{(\nu)}$ be the function which is $\sigma(h)\nu$ at $h \in H$ and 0 off H . Then $\nu \mapsto (A\delta_e^{(\nu)})(x)$ is a linear operator on $\mathcal{H}_0$, which we denote by A_x .

Lemma 3. *The A_x determine A .*

Proof. Let $\delta_y^{(\nu)}$ be $\sigma(h)v$ at hy and 0 off Hy . Then $\delta_y^{(\nu)} = \pi(y^{-1}) \cdot \delta_e^{(\nu)}$, and therefore

$$(A\delta_y^{(\nu)}(x)) = A\pi(y^{-1})\delta_e^{(\nu)}(x) = \pi(y^{-1})(A\delta_e^{(\nu)})(x) = A\delta_e^{(\nu)}(xy^{-1}),$$

or

$$(1) \quad A\delta_y^{(\nu)}(x) = A_{xy^{-1}}v.$$

Hence the A_x determine A on a dense subset of $\mathcal{H}$, and this proves the lemma.

Note. We also have (from (1))

$$(2) \quad A\delta_y^{(\nu)} = \sum_{x \in H/G} \delta_x^{(w)}, \quad \text{where } w = A_{xy^{-1}}v.$$

For the next theorem, we need the following combinatorial lemma.

Lemma 4. *A group G cannot be covered by finitely many cosets of finitely many subgroups of infinite index.*

Proof. We may assume that all cosets are right cosets, since $xH = (xHx^{-1})x$. We use induction on the number n of subgroups involved. Assume the theorem for n , and suppose that the cosets are cosets of $H_1, \dots, H_{n+1}$. Let $K_j = K_j \cap H_{n+1}$. We may assume that if $1 \leq j \leq n$, $H_j \cap H_{n+1}$ has infinite index in H_{n+1} ; otherwise, H_{n+1} is covered by finitely many cosets of H_j , and therefore we can replace H_{n+1} -cosets with H_j -cosets. We may also assume (perhaps by translating all cosets by some $x \in G$) that H_{n+1} itself is not one of the cosets in the union. Hence H_{n+1} is covered by finitely many cosets of $H_1, \dots, H_n$, and therefore H_{n+1} is the union of finitely many cosets of $K_1, \dots, K_n$. The inductive hypothesis says that this is impossible, and the lemma follows.

Theorem 2. *If $A = \{A_x\} \in \text{Hom}(\pi, \pi)$, then $A_x = 0$ unless $x \in Q_H$. For $x \in Q_H$ and $h, h' \in H$, we have $A_{h_x h'} = \sigma(h)A_x\sigma(h')$, and A_x intertwines σ and σ^x (where $\sigma^x(x^{-1}hx) = \sigma(h)$) on H_x .*

Proof. $A_{h_x}^{(\nu)} = A\delta_e^{(\nu)}(hx) = \sigma(h)A\delta_e^{(\nu)}(x) = \sigma(h)A_x(v)$, because $A\delta_e^{(\nu)} \in \mathcal{H}$. Furthermore, if $h \in H$, $\delta_h^{(\nu)}$ and $\delta^{(w)}$, where $w = \sigma(h^{-1})v$, are both concentrated on H and agree at h ; thus they are equal. But

$$A\delta_h^{(\nu)}(x) = A_{xh^{-1}}(v) \quad \text{and} \quad A\delta_e^{(w)}(x) = A_x\sigma(h^{-1})v.$$

Thus $A_{h_x} = \sigma(h)A_x$ and $A_{xh'} = A_x\sigma(h')$ for $h, h' \in H$. If $h = x^{-1}h_0x$ ($h, h_0 \in H$), then

$$\sigma^x(h)A_x = \sigma(h_0)A_x = A_{h_0x} = A_{xh} = A_x\sigma(h),$$

which proves the claim about A_x being an intertwining operator. So all that we need to show is that if $A_x \neq 0$, then $x \in Q_H$.

Assume $A_x \neq 0$. For each unit vector $v \in \mathcal{H}_0$, there is an element $h \in H$ such that $A_x\sigma(h)v \neq 0$, since the unit vectors $\sigma(h)v$ span $\mathcal{H}_0$. Thus the open subsets $U_h = \{v: \|v\| = 1, A_x\sigma(h)v \neq 0\}$ cover the unit ball of $\mathcal{H}$. We may pick a finite set $\{h_1, \dots, h_n\} \subseteq H$, plus a number $\alpha > 0$, such that for any unit vector v , some vector $A_x\sigma(h_i)v$ has length $\geq \alpha$. Now suppose that $[H: H_x] = \infty$. Then we can pick elements $k_1, k_2, \dots$ of H such that if $i \neq j$, the cosets $H_x h_1 k_i, \dots, H_x h_n k_i$ are distinct from the cosets $H_x h_1 k_j, \dots, H_x h_n k_j$. (This follows from the fact that a group cannot be covered by finitely many cosets of subgroups of infinite index.) We know that for each k_i , one of the vectors $A_x\sigma(h_1 k_i)v, \dots, A_x\sigma(h_n k_i)v$ has length $\geq \alpha$. Let it be $A_x\sigma(h_{j(i)} k_i)v$. As the elements $xh_{j(i)} k_i$ are in distinct right H -cosets,

$$\|A\delta_e^{(v)}\| \geq \sum_{i=1}^{\infty} \|A_{xh_{j(i)} k_i} v\| \geq \sum_{i=1}^{\infty} \alpha^2 = \infty,$$

which is impossible. Hence $[H: H_x] < \infty$.

Finally, we compute

$$\begin{aligned} A^*: \langle \delta_x^{(w)}, A^* \delta_e^{(v)} \rangle &= \langle A \delta_x^{(w)}, \delta_e \rangle = \langle A \delta_x^{(w)}(e), v \rangle \\ &= \langle A_{x^{-1}} w, v \rangle = \langle w, (A_{x^{-1}})^* v \rangle = \langle \delta_x^{(w)}, B \delta_e^{(v)} \rangle, \end{aligned}$$

where $B_x = (A_{x^{-1}})^*$. Hence

$$(3) \quad (A^*)_x = (A_{x^{-1}})^*.$$

But $A^* \in \text{Hom}(\pi, \pi)$; hence $[H: H_{x^{-1}}] < \infty$. This completes the proof. Note, incidentally, that A_x also intertwines $\sigma^{x^{-1}}$ and σ on $H_{x^{-1}}$.

We now prove a converse of sorts to Theorem 2. First, a lemma.

Lemma 5. *Let $\{T_j\}_{j=1}^m$ be any sequence of operators on a finite-dimensional inner product space V . There is a constant c (depending only on $\dim V$) and a vector $v \in V$ such that*

$$c \sum_{j=1}^m \|T_j v\|^2 \geq \sum_{j=1}^m \|T_j\|^2.$$

Proof. Let $v_1, \dots, v_n$ be an orthonormal basis for V , and let $\|\cdot\|$ be the trace norm on operators ($\|\cdot\|^2 = \text{Tr}(T^* T)$). As all norms on $\mathcal{B}(V)$ are equivalent, and as

$$\sum_{i=1}^n \|T_j v_i\|^2 = \sum_{i=1}^n \langle T_j^* T_j v_i, v_i \rangle = \|T_j\|^2,$$

there is a constant c_0 such that $c_0 \sum_{i=1}^n \|T_j v_i\|^2 \geq \|T_j\|^2$ for all j (c_0 depends only on $n = \dim V$). Summing, we see that

$$c_0 \sum_{j=1}^m \sum_{i=1}^n \|T_j v_i\|^2 \geq \sum_{j=1}^m \|T_j\|^2,$$

and hence

$$\exists i: c_0 n \sum_{j=1}^m \|T_j v_i\|^2 \geq \sum_{j=1}^m \|T_j\|^2.$$

Let $c = c_0 n$.

Theorem 3. Suppose that $x \in Q_H$, and that A_x intertwines σ and σ^x on H_x . Define $A = \{A_y\}$, where $A_y = \sigma(h)A_x\sigma(h')$ if $y = hxb'$ and 0 otherwise. Then $A \in \text{Hom}(\pi, \pi)$. The A 's span a weakly dense subspace in $\text{Hom}(\pi, \pi)$.

Proof. Let

$$(A^0 f)(y) = \sum_{h \in H_x^{-1}H} \sigma(h^{-1})A_x f(x^{-1}hy).$$

The sum is finite and well defined; if we replace h by $h_1 h$, where $h_1 \in H_x$ (so that $h_1 \in H$, $h_1 = xh_2x^{-1}$ with $h_2 \in H$), then

$$\begin{aligned} \sigma(h_1 h)^{-1} A_x f(x^{-1}h_1 hy) &= \sigma(h^{-1})\sigma(h_1^{-1})A_x f(h_2 x^{-1}hy) \\ &= \sigma(h)^{-1}\sigma(h^{-1})A_x \sigma(h_2) f(x^{-1}hy) \\ &= \sigma(h^{-1})\sigma(h_1^{-1})\sigma^x(h_2)A_x f(x^{-1}hy) \\ &= \sigma(h^{-1})A_x f(x^{-1}hy), \end{aligned}$$

so that each term is independent of the coset representative taken. Also, if $h' \in H$,

$$\begin{aligned} (A^0 f)(h'y) &= \sum_h \sigma(h^{-1})A_x f(x^{-1}hh'y) \\ &= (h \mapsto hh'^{-1}) \sum_h \sigma(h')\sigma(h^{-1})f(x^{-1}hy) = \sigma(h')A^0 f(y). \end{aligned}$$

Hence $A^0: \mathcal{H} \rightarrow \mathcal{H}$. Clearly $A^0 \in \text{Hom}(\pi, \pi)$. Finally, we compute $(A^0)y$. Since

$$A_y^0(v) = \sum_h \sigma(h)A_x \delta_e^{(v)}(x^{-1}hy),$$

from (1), and since this sum is 0 unless $x^{-1}hy \in H$ for some $h \in H$, $A_y^0 = 0$ for $y \notin HxH$. If $y = h_1 x h_2$, then we may assume that h_1 is one of the coset representatives for $Hx^{-1}H$; the term involving h_1 will be the only nonzero one, and

$$A_{h_1 x h_2}^0(v) = \sigma(h_1) A_x \delta_e^{(v)}(h_2) = \sigma(h_1) A_x \sigma(h_2) v.$$

Hence $A^0 = A$.

Now let $B = \{B_y\}$ be an arbitrary element of $\text{Hom}(\pi, \pi)$. Then for every $v \in \mathcal{H}_0$ of length 1, $\|B \delta_e^{(v)}\| \leq \|B\|$; hence $\sum_{y \in H \setminus G} \|B_y(v)\|^2 \leq \|B\|^2$. Lemma 5 says that $\sum_{y \in H \setminus G} \|B_y\|^2 < \infty$. In particular, B_y is concentrated on countably many right H -cosets of G . If

$$f = \sum_{j=1}^n \alpha_j \delta_{x_j}^{(v_j)}, \quad \text{then } (Bf)(z) = \sum_{j=1}^n \alpha_j B_{zx_j^{-1}}(v_j).$$

Now let

$$f = \sum_{j=1}^{\infty} \alpha_j \delta_{x_j}^{(v_j)} \in L^2(\mathcal{H})(G), \quad f_n = \sum_{j=1}^n \alpha_j \delta_{x_j}^{(v_j)}.$$

Because $f_n \rightarrow f$, $Bf_n \rightarrow Bf$; hence $\sum_{j=1}^{\infty} \alpha_j B_{zx_j^{-1}}(v_j)$ converges. If we pick a basis of orthonormal vectors in V , we see similarly that the sum converges for each component. Moreover, it converges for each component regardless of how the terms are rearranged, since we can label the x_j in any order. Thus the sum for each component converges absolutely. Now order the double cosets Hy_jH on which B_y is nonzero, and let $B_y^{(n)} = B_y$ if $y \in \bigcup_{j=1}^n Hy_jH$, and 0 otherwise. Then $B_y^{(n)}$ is in the space spanned by the A 's, and if $f = \sum_{j=1}^{\infty} \alpha_j \delta_{x_j}^{(v_j)}$,

$$(B^{(n)}f)(z) = \sum_{j=1}^{\infty} \alpha_j B_{zx_j^{-1}}^{(n)}(v_j) \rightarrow Bf(z) \quad \text{as } n \rightarrow \infty.$$

Hence $B^{(n)} \rightarrow B$ strongly (a fortiori weakly).

Corollary 1. π is irreducible $\Leftrightarrow$ for all $x \in Q_H$, $x \in H$, $\pi^{(x)}$ and π are disjoint on H_x .

Corollary 2. Let $\tau = \text{Ind } \sigma_{H \rightarrow Q_H}$. Then $\text{Hom}(\pi, \pi) \cong \text{Hom}(\tau, \tau)$.

As an example, let G be the free group on two generators x and y , and let H be the subgroup generated by x . Then $Q_H = H$, and it follows, for instance, that the "regular" representation of G on $H \setminus G$ is irreducible.

If G is finite (so that $Q_H = G$), Corollary 1 amounts to Mackey's generalization of a theorem of Shoda [1, Theorem 6].

Next, we look briefly at the center of $\text{Hom}(\pi, \pi)$. We shall assume (as we may without loss of generality) that $G = Q_H$.

Theorem 4. $B = \{B_y\}$ is central in $\text{Hom}(\pi, \pi) \Leftrightarrow$ for all $x, z \in G$ and all A_x intertwining σ and σ^* on H_x ,

$$\sum_{h \in H_x \setminus H} B_{zh^{-1}x^{-1}} A_x \sigma(h) = \sum_{h \in H_x \setminus H} \sigma(h) A_x B_{x^{-1}h^{-1}z}.$$

Proof. Let $A = \{A_y\}$ be the operator constructed from A_x as in Theorem 3. We know that B is central $\Leftrightarrow AB = BA$, all such $A \Leftrightarrow AB\delta_e^{(\nu)} = BA\delta_e^{(\nu)}$, all such A . So we compute

$$BA\delta_e^{(\nu)}(y) = B \left(\sum_{y \in H \setminus HxM} \delta_y^{(wy)} \right) (z) = \sum_{y \in H \setminus HxH} B_{zy^{-1}} A_y(v) \quad (w_y = A_y(v)).$$

But to get the coset representatives, we may let $y = xh$, h running through coset representatives of $H_x \setminus H$; thus

$$(4) \quad BA\delta_e^{(\nu)}(z) = \sum_{h \in H_x \setminus H} B_{zh^{-1}x^{-1}} A_{xh}(v) = \sum_{h \in H_x \setminus H} B_{zh^{-1}x^{-1}} A_x \sigma(h)v,$$

$$AB\delta_e^{(\nu)}(z) = A \left(\sum_{y \in H \setminus G} \delta_y^{(vy)}(z) \right) = \sum_{y \in H \setminus G} A_{zy^{-1}} B_y(v) \quad (v_y = B_y(v)).$$

Again, we must have $zy^{-1} = hxb^{-1}$, or $y = h'^{-1}x^{-1}h^{-1}z$; to get coset representatives, let $h' = e$ and h run through $H_x \setminus H$. One gets

$$(5) \quad AB\delta_e^{(\nu)}(z) = \sum_{h \in H_x \setminus H} A_{hx} B_{x^{-1}h^{-1}z}(v) = \sum_{h \in H_x \setminus H} \sigma(h) A_x B_{x^{-1}h^{-1}z}(v).$$

Now combine (4) and (5).

Corollary 1. If x normalizes H and $\sigma = \sigma^x$ on H , then one has

$$A_x^{-1} B_z A_x = B_{x^{-1}zx}$$

if B is central.

Corollary 2. If $\dim \mathcal{H}_0 = 1$, the condition of Theorem 4 becomes

$$\sum_{h \in H_x \setminus H} \sigma(h) B_{zh^{-1}x^{-1}} = \sum_{h \in H_x \setminus H} \sigma(h) B_{x^{-1}h^{-1}z}.$$

These sums are unpleasant to deal with; in one case, where they disappear, we can say a little.

Theorem 5. Let $N_H = \{x \in G: x \text{ normalizes } H \text{ and } \sigma^x = \sigma\}$, $M_H =$

$\{x \in N_H: \text{the } N_H\text{-conjugates of } x \text{ lie in finitely many } H\text{-cosets}\}$. If $[N_H = M_H] = \infty$, π is Type II.

Proof. We may assume $G = N_H$ without loss of generality; in particular, H is normal. From this, it follows easily that M_H is indeed a group. Now one can mimic Mautner's proof in [3] that if G is a discrete group, G_0 is the subgroup of elements with finitely many conjugates, and $[G: G_0]$ is infinite, then the regular representation of G is Type II. This proves the theorem.

Theorem 6. *In the above situation, let M'_H = commutator subgroup of M_H . If $[M'_H: H \cap M'_H]$ is infinite, π is Type II.*

Proof. From Theorem 5, we may as well assume $[N_H: M_H] < \infty$; then we may as well assume that $N_H = M_H$, since inducing from M_H to N_H will change nothing. Suppose that π has a Type I_n factor for some $n \geq 1$. The argument in Theorem 1 of [4] now goes through essentially unchanged to show that the commutator subgroup $[M_H/H, M_H/H]$ is finite, and this implies the result.

Theorem 7. *Suppose $G = N_H$ and π is Type II. Then $[N_H = M_H] = \infty$ or $[M'_H: H \cap M'_H]$ is infinite.*

Proof. Pick $x \in G$, and let A_x implement the equivalence of σ with $\sigma^{(x)}$. Let C_1 project $\mathcal{H}$ onto $\mathcal{H}_0$ by evaluation at x_1 and C_0 by evaluation at e . Then $C_0 \in \text{Hom}(\pi/H, \sigma)$, and $C_1 \in \text{Hom}(\pi/H, \sigma^x)$, since

$$\begin{aligned} \pi(h)\delta_x^{(\nu)}(x) &= \delta_x^{(\nu)}(xh) = \delta_x^{(\nu)}(xhx^{-1} \cdot x) \\ &= \sigma(xhx^{-1})(\delta_x^{(\nu)}(x)) = \sigma^{(x)}(h)(\delta_x^{(\nu)}(x)). \end{aligned}$$

Thus $A_x C_0$ is a multiple of C_2 , and we may change A_x if necessary so that $A_x C_0 = C_1$. Note that if we change x to hx , the correct choice for A_{hx} is $\sigma(h)A_x$.

Now let P be the operator given by

$$P(f)y = \frac{1}{m} \sum_{x \in M \setminus HM_M} A_{x^{-1}} f(xy),$$

where $m = [M'_H: H]$. $P \in \text{Hom}(\pi, \pi)$; in fact, $P = (P_x)$, where $P_x = A_x/m$ for $x \in M'_H H$, and = 0 otherwise. Since $A_x A_y = A_{xy}$ (thanks to our normalization), $P^2 = P$, and Theorem 4 implies that P is central. Now the last part of the proof of Theorem 1 of [4] goes through (with P for f) without any trouble.

Theorems 5, 6, and 7 are more or less copies of the corresponding results for $H = \{e\}$. In case H is not normal, the situation seems considerably more obscure.

Added in proof. Lemma 4 is found in B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. 29 (1954), 236–248. MR 15, 931.

Corollary 1 was proved by Mackey in the case where $\dim \sigma = 1$. See [1].

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