UNIQUE HAHN-BANACH EXTENSIONS AND KOROVKIN'S THEOREM

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ABSTRACT. This paper characterizes in terms of weak topologies those bounded linear functionals on a subspace which have unique Hahn-Banach extensions to the whole linear normed space. The relationship to the Choquet boundary is discussed, and a Korovkin type theorem is obtained.

We begin with a different proof of Korovkin's theorem in order to motivate that which follows.

Theorem 1 (Korovkin [2]). *Suppose $\{L_n\}$ is a sequence of positive linear functionals on $C[0, 1]$ satisfying $L_n(1) \rightarrow 1$, $L_n(I) \rightarrow c$, and $L_n(I^2) \rightarrow c^2$. (Here I is the identity function.) Then $L_n(f) \rightarrow f(c)$ for all $f \in C[0, 1]$.*

Proof of Theorem 1. Identify C^* with the regular Borel measures on $[0, 1]$. Identify L_n with a regular positive measure by $L_n(f) = \int f d\mu_n$. Any subnet μ_α of $\{\mu_n\}$ has a further subnet μ_γ converging $w(C^*, C)$ to some μ , and it follows that $\mu_\gamma(1) \rightarrow \mu(1) = \|\mu\|$. In particular $\mu(1) = 1$, $\mu(I) = c$, and $\mu(I^2) = c^2$. Then μ must be ξ_c , the point evaluation measure, for suppose $\mu(\{c\}) < 1$; then there is an open interval J containing c with $\mu(J) < 1$. Let $g(x) = (x - c)^2$ and $\tau = \min \int g(x) d\mu > 0$. Then

$$0 = \int_0^1 (1 - c)^2 d\mu \geq \int_J (1 - c)^2 d\mu \geq \tau \mu(J) = \tau(1 - \mu(J)) > 0,$$

a contradiction. Since every subnet of $\{\mu_n\}$ has a further subnet converging $w(C^*, C)$ to ξ_c , then $\mu_n \rightarrow \xi_c$ in $w(C^*, C)$, hence $L_n f = \int f d\mu_n \rightarrow \int f d\xi_c = f(c)$ for all $f \in C[0, 1]$.

While this proof is admittedly less elementary than Korovkin's, it makes it clear that the heart of the matter is the fact that the point evaluation func-

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tional ξ_c is the only Hahn-Banach extension of its restriction to the subspace $\text{span}(1, l, l^2)$. This, of course, can be viewed as resulting from the fact that the Choquet boundary of $\text{span}(1, l, l^2)$ is $[0, 1]$. Wulbert [7], [8], using this observation and the fact that point evaluations are the extreme points in the unit ball of C^* , has defined a generalized Choquet boundary $cb(P)$ for weakly separating subspaces P of a linear normed space, and he proved a nice Korovkin theorem for such subspaces. In this paper we take the position that the central issue is not extreme points, but unique Hahn-Banach extensions.

If (X, T_1, T_2) is a space with two topologies, we shall say the topologies T_1 and T_2 are equivalent at the point x if they have equivalent neighborhood bases at x . This can happen if and only if each net x_δ in X converging to x in T_1 converges to x in T_2 and vice versa. Also recall that if $S \subset X$, x_δ is a net in S , and $x \in S$, then $x_\delta \rightarrow x$ in the relative T_1 topology on S if and only if $x_\delta \rightarrow x$ in (X, T_1) [6].

In what follows, if X is a linear normed space, we denote by X^* the space of continuous linear functionals on X , by S^* the unit ball in X^* , and by $w(X^*, X)$ the weak-star topology on X^* .

Definition. If X is a linear normed space and M is a subspace, we define the set $K(M)$ by $K(M) = \{x^* \in X^* \mid \|x^*\| = 1 \text{ and such that } x^*|_M = y^*|_M \text{ implies } x^* = y^* \text{ if } \|y^*\| \leq 1\}$. Note that if $X = C[0, 1]$, M is a subspace containing the constants and separating points of $[0, 1]$, and the points of the Choquet boundary $B(M)$ are identified with their point evaluation functionals, then $B(M) \subset K(M)$. Examples with other subspaces M follow.

(i) $M = \text{span}(1)$. The value $\mu(1)$ is the μ measure of $[0, 1]$, and it is clear that any μ with $\|\mu\| = 1$ has many different Hahn-Banach extensions, so $K(M) = \Phi$.

(ii) $M = \{f \mid f(1/2) = (f(0) + f(1))/2\}$ (Phelps [3]). Here $B(M) = \{\xi_\alpha \mid 0 \leq \alpha \leq 1, \alpha \neq 1/2\}$. Here $B(M) \subset K(M)$ properly since any measure μ with $\|\mu\| = 1$ having the property that there is an open interval $J \subset [0, 1]$ with $1/2 \in J$ and $|\mu|(J) = 0$ is in $K(M)$. Note that $\xi_{1/2} \notin K(M)$.

(iii) If M is dense, then $K(M) = \{x^* \in S^* \mid \|x^*\| = 1\}$.

We now formulate a theorem characterizing the points of $K(M)$.

Theorem 2. A point $x^* \in S^*$ with $\|x^*\| = 1$ is in $K(M)$ if and only if the topologies induced on S^* by $w(X^*, M)$ and $w(X^*, X)$ are equivalent at x^* .

Proof of Theorem 2. Suppose $x^* \in K(M)$. Suppose x_α^* is a net in S^* converging to x^* in relative $w(X^*, M)$. Any subnet x_β^* has a $w(X^*, X)$ (hence

relative $w(X^*, X)$ convergent subnet x_γ^* , say $x_\gamma^* \rightarrow y^*$. It follows that $y^* \in S^*$ and $y^*|M = x^*|M$. Hence $y^* = x^*$ since $x^* \in K(M)$ and $x_\alpha^* \rightarrow x^*$ in relative $w(X^*, X)$. Conversely, suppose the two topologies are equivalent at $x^* \in S^*$, where $\|x^*\| = 1$. Suppose $y^* \in S^*$ and $y^*|M = x^*|M$. Define a net $x_\alpha^* = y^*$. Then $x_\alpha^* \rightarrow y^*$ relative $w(X^*, M)$, hence in relative $w(X^*, X)$, hence in $w(X^*, X)$. Similarly $x_\alpha^* \rightarrow x^*$ relative $w(X^*, M)$, hence in $w(X^*, X)$. This implies that $x^* = y^*$, so $x^* \in K(M)$ and the proof is complete.

In [7], Wulbert proves a theorem similar to the "only if" part of the last theorem for weakly separating subspaces M and the corresponding $cb(M)$. In this case $cb(M) \subset K(M)$; for suppose $x^* \in cb(M)$; then x^* is extreme in $S(M^*)$. Suppose $y^* \in S^*$ with $y^*|M = x^*|M$. If y^* is extreme in S^* , then $y^* = x^*$ since M is weakly separating. In fact, y^* must be extreme in S^* , for if not, then y^* and x^* are in the face $F = \{z^* \mid \|z^*\| = 1 \text{ and } z^*|M = x^*|M\}$, and the Krein-Milman theorem implies the existence of an extreme point $w^* \neq x^*$ in F . This contradicts the fact that M is weakly separating. This shows $x^* \in K(M)$ and $cb(M) \subset K(M)$. We also observe that if $x^* \in K(M)$ and $x^*|M$ is extreme in $S(M^*)$, then x^* is extreme in S^* ; for suppose $x^* = ty^* + (1-t)z^*$, $y^*, z^* \in S^*$, $0 < t < 1$. Restricting this to M gives $x^*|M = y^*|M = z^*|M$ since $x^*|M$ is extreme in $S(M^*)$. But then $x^* = y^* = z^*$, since $x^* \in K(M)$, hence x^* is extreme in S^* .

We now formulate a Korovkin type theorem.

Theorem 3. *If L_α is a net of operators in $B[X, X]$ with $\|L_\alpha\| \leq 1$, and if $\|L_\alpha x - x\| \rightarrow 0$ for all x in a subspace M , then $x^* L_\alpha x \rightarrow x^* x$ uniformly on all $w(X^*, X)$ compact subsets of $K(M)$ for each x in X .*

Proof of Theorem 3. If not, there exist $\epsilon > 0$, a $w(X^*, X)$ compact set $D \subset K(M)$, $x \in X$, a subnet L_β of L_α , and a net $x_\beta^* \in D$ such that

$$(*) \quad |x_\beta^* L_\beta x - x_\beta^* x| \geq \epsilon.$$

By taking a further subnet, we may assume $x_\beta^* \rightarrow x^* \in K(M)$ in $w(X^*, X)$. If $y \in M$, then

$$\begin{aligned} |x_\beta^* L_\beta y - x^* y| &\leq |x_\beta^* L_\beta y - x_\beta^* y| + |x_\beta^* y - x^* y| \\ &\leq \|x_\beta^*\| \|L_\beta y - y\| + |x_\beta^* y - x^* y| \rightarrow 0. \end{aligned}$$

But since $x^* \in K(M)$ and $\|x_\beta^* L_\beta\| \leq \|x_\beta^*\| \|L_\beta\| \leq 1$, this implies $x_\beta^* L_\beta(x) \rightarrow x^*(x)$ for all $x \in X$. This together with the fact that $x_\beta^*(x) \rightarrow x^*(x)$ for all $x \in X$ contradicts inequality (*) above, and completes the proof.

While this argument is much the same as Wulbert's [8], it should be noted we have no assumptions on M other than it is a subspace, and $cb(M) \subset K(M)$ can be a proper containment when $cb(M)$ exists.

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