ON PRIME IDEALS WITH GENERIC ZERO $x_i = t^{n_i}$

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ABSTRACT. Let $n_i, 1 \leq i \leq r, r \geq 3$, be natural numbers such that $(n_1, \dots, n_r) = 1$ and $n_i = \sum_{j=1}^r z_j n_j, z_j$ nonnegative integers, implies $z_j = 0, j \neq i$, and $z_i = 1$. It is shown that for prime ideals with generic zero $x_i = t^{n_i}$ and $r \geq 4$, arbitrary large finite minimal sets of generators exist.

Let $Z = Z^-$ denote the nonnegative integers, let n_i be a natural number, $1 \leq i \leq r, r \geq 3$, and let $\langle n_1, \dots, n_r \rangle$ be the additive semigroup $\{z | z = \sum_{i=1}^r z_i n_i, z_i \in Z - Z^-\}$. Assume further $(n_1, \dots, n_r) = 1$ and $n_i \notin \langle n_1, \dots, \hat{n}_i, \dots, n_r \rangle$. Let K be an arbitrary field and $P(n_1, \dots, n_r)$ the prime ideal $\{f(x_1, \dots, x_r) | f(x_1, \dots, x_r) \in K[x_1, \dots, x_r], f(t^{n_1}, \dots, t^{n_r}) = 0, t \text{ transcendental over } K\}$. Let $|P(n_1, \dots, n_r)| < \infty$ be the smallest number of generators of $P(n_1, \dots, n_r)$, and let $P(r) = \text{l.u.b. } \{z | z = |P(n_1, \dots, n_r)|\}$. In [2] it is shown that for $r = 3, P(3) = 3$ and $|P(n_1, n_2, n_3)| = 2$ iff $\langle n_1, n_2, n_3 \rangle$ is symmetric. The purpose of this note is to prove the following statement.

Theorem. $r \geq 4 \Rightarrow P(r) = \infty$.

The proof is by induction on r , hence initially $r = 4$. Let $q_1 = q_2 + 1, q_2$ even, $q_2 \geq 4, d_1 = q_2 - 1$. Set $n_1 = q_1 q_2, n_2 = q_1 d_1, n_3 = q_1 q_2 + d_1, n_4 = q_2 d_1. (n_1, n_2, n_3, n_4) = 1, n_h \notin \langle \hat{n}_h, n_i, n_j, n_k \rangle,$

$$A_1 = \{f_\mu | f_\mu = x_1^{\mu-1} x_3^{q_2-\mu} - x_2^{q_2-\mu} x_4^{\mu+1}, 1 \leq \mu \leq q_2\} \subseteq P(n_1, n_2, n_3, n_4) = P,$$

$$\{g_1 = x_1^{d_1} - x_2^{q_2}, g_2 = x_3 x_4 - x_2 x_1\} \subseteq P \text{ follows readily.}$$

Lemma 1. $\alpha_h n_h \in \langle \hat{n}_h, n_i, n_j, n_k \rangle, \alpha_h \text{ minimal} \Rightarrow \alpha_1 = d_1, \alpha_2 = q_2, \alpha_3 = d_1, \text{ and } \alpha_4 = q_1.$

Proof. $\nu_1 q_1 q_2 = \mu_2 q_1 d_1 + \mu_3 (q_1 q_2 + d_1) + \mu_4 q_2 d_1 \Rightarrow \nu_1 > \mu_3, \text{ and}$

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$d_1 | \nu_1 - \mu_3 \Rightarrow \nu_1 \geq d_1$. The other proofs are similar.

Let

$$A_2 = \{f | f = x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3}, \nu_1, \mu_3 < d_1, f \in P\},$$

$$A = A_1 \cup A_2 \cup \{g_1, g_2\}, A' = A \cup \{f | f = x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3}, f \in P\},$$

and let (A) and (A') be the ideals generated by the respective sets.

Lemma 2. $(A') = P \Rightarrow (A) = P$.

Proof. Replace $x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3} \in A'$, $\nu_1, \mu_3 \geq d_1$, by

$$x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3} - x_1^{\nu_1-d_1} x_4^{\nu_4} f_{q_2} = x_1^{\nu_1-d_1} x_4^{\nu_4+q_1} - x_2^{\mu_2} x_3^{\mu_3}.$$

$$f = x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3} \in A', \nu_1 < d_1, \mu_3 \geq d_1 \Rightarrow$$

$$x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3} + x_2^{\mu_2} x_3^{\mu_3-d_1} f_1 = x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2+d_1} x_3^{\mu_3-d_1} x_4^2.$$

$$\nu_1 < d_1 \Rightarrow \nu_4 > 2 \Rightarrow f' = x_1^{\nu_1} x_4^{\nu_4-2} - x_2^{\mu_2+d_1} x_3^{\mu_3-d_1} \in A'. \text{ Replace } f \text{ by } f'.$$

Lemma 3. $(A') = P$.

Proof. By [2, Proposition 1.4], $P(n_1, \dots, n_r)$ is generated by

$$F(\nu, \mu) = x_1^{\nu_1} \dots x_r^{\nu_r} - x_1^{\mu_1} \dots x_r^{\mu_r}, \quad \partial(F(\nu, \mu)) = \sum_{i=1}^r \nu_i n_i = \sum_{i=1}^r \mu_i n_i,$$

ν_i or $\mu_i = 0$, $1 \leq i \leq r$. The existence of $f \in (A')$ $F(\nu, \mu) - f = \prod_{i=1}^4 x_i^{\delta_i} g$, $g = 0$ or $g = F(\nu', \mu')$, $\partial(F(\nu', \mu')) < \partial(F(\nu, \mu))$ will prove the lemma.

If in $F(\nu, \mu)$, $\nu_3 = \mu_3 = 0$, then $F(\nu, \mu) \in P \cap K[x_1, x_2, x_4]$. $\langle n_1, n_2, n_4 \rangle$ symmetric and $[2] \Rightarrow P \cap K[x_1, x_2, x_4] = (g_1, f_{q_2}) \subseteq (A')$. Hence assume

$\nu_3 \neq 0$.

(1) $\nu_1 = \nu_2 = \nu_4 = 0 \Rightarrow \nu_3 \geq d_1$. $\mu_2 = \mu_4 = 0 \Rightarrow \mu_1 \geq d_1$ and $f = -x_1^{\mu_1-d_1} g_1 + x_3^{\nu_3-d_1} f_1$. μ_2 or $\mu_4 \neq 0 \Rightarrow f = x_3^{\nu_3-d_1} f_1$.

(2) Exactly one $\nu_j \neq 0$, $j \neq 3$. $\nu_2 \neq 0 \Rightarrow F(\nu, \mu) \in (A')$. $\nu_4 \neq 0 \Rightarrow f = x_3^{\nu_3-1} x_4^{\nu_4-1} g_2$. Let $\nu_1 \neq 0$. $\nu_3 \geq d_1 \Rightarrow f = x_1^{\nu_1} x_3^{\nu_3-d_1} f_1$.

$$\nu_1 \geq d_1 \Rightarrow f = x_1^{\nu_1 - d_1} x_3^{\nu_3} g_1 \quad \text{or} \quad f = x_1^{\nu_1 - d_1} x_3^{\nu_3} f_{q_2}.$$

$$\nu_1, \nu_3 < d_1 \Rightarrow F(\nu, \mu) = x_1^{d_1 - \sigma} x_3^{\nu_3} - x_2^{\mu_2} x_4^{\mu_4},$$

$$\sigma \text{ and } \nu_3 < d_1 \Rightarrow (d_1 - \sigma)q_1q_2 + \nu_3(q_1q_2 + d_1) = \mu_2q_1d_1 + \mu_4q_2d_1$$

$$\Rightarrow d_1 | d_1 - \sigma + \nu_3 \Rightarrow \sigma = \nu_3 \Rightarrow f \in A_1.$$

$$(3) \text{ Exactly one } \nu_j = 0, j \neq 3. f \text{ is } -x_1^{\mu_1 - d_1} g_1, x_2^{\mu_2 - q_2} g_1 \text{ or } x_4^{\mu_4 - q_1} f_{q_2},$$

according to which $\mu_j \neq 0, 1 \leq j \leq 4, j \neq 3$. Thus $(A) = P$.

Let P^e be the extension of P in $K[x_1, x_2, x_3, x_4]$, the formal ring of power series.

Lemma 4. $\bar{f}_\mu \in P^e/P^e(x_1, x_2, x_3, x_4), 1 \leq \mu \leq q_2$, are linearly independent over K .

Proof. Following a procedure similar to [1, p. 47], let ϕ be the substitution homomorphism $x_1 \rightarrow T, x_2 \rightarrow S^{q_1}, x_3 \rightarrow TS, x_4 \rightarrow S^{q_2}$. Then $\phi(g_1) = T^{d_1} - S^{q_1q_2} = w, \phi(f_\mu) = S^{q_2 - \mu} w, 1 \leq \mu \leq q_2, \phi(g_2) = 0$.

$$f = x_1^{\nu_1} x_4^{\nu_4} - x_2^{\mu_2} x_3^{\mu_3} \in A_2 \Rightarrow \phi(f) = T^{\nu_1} S^{\nu_4q_2} - S^{\mu_2(q_2+1)+\mu_3} T^{\mu_3}.$$

Since

$$\begin{aligned} \nu_1q_1q_2 + \nu_4q_2d_1 &= \mu_2q_1d_1 + \mu_3(q_1q_2 + d_1) \\ \Leftrightarrow (\nu_1 - \mu_3)q_1q_2 &= (\mu_2q_1 + \mu_3 - \nu_4q_2)d_1 \Leftrightarrow \nu_1 = \mu_3 \end{aligned}$$

and $\nu_4q_2 = \mu_2q_1 + \mu_3, \phi(f) = 0$. Applying ϕ to $\sum_{\mu=1}^{q_2} c_\mu f_\mu \in P^e(x_1, x_2, x_3, x_4)$, $c_\mu \in K$, gives $\sum_{\mu=1}^{q_2} c_\mu S^{q_2 - \mu} \in \phi(x_1, x_2, x_3, x_4)$. Since any nonzero polynomial $f \in K[S] \cap \phi(x_1, x_2, x_3, x_4)$ has degree $\geq q_2, c_\mu = 0, 1 \leq \mu \leq q_2$.

Corollary 1. $P = (\pi_1, \dots, \pi_n) \Rightarrow n \geq q_2$.

Corollary 2. $P(4) = \infty$.

Finally assume $P(r) = \infty, r \geq 4$. Let $|P(n_1, \dots, n_r)| = m$.

Since $P(n_1, \dots, n_r)$ is homogeneous in the $\langle n_1, \dots, n_r \rangle$ -graded ring $K[x_1, \dots, x_r]$, and since K is a field, by [3, Proposition 2 and Theorem 2], any two reduced sets of homogeneous generators are equal in number and minimal. Thus $\exists F(\nu_i, \mu_i), 1 \leq i \leq m$, generating $P(n_1, \dots, n_r)$. Set $n'_i =$

$2n_i$, $1 \leq i \leq r$, and let n'_{r+1} be a prime $> 2\partial(F(\nu_i, \mu_i))$, $1 \leq i \leq m$. Again, since $P(n'_1, \dots, n'_{r+1})$ is homogeneous in the $\langle n'_1, \dots, n'_{r+1} \rangle$ -graded ring $K[x_1, \dots, x_{r+1}]$, and since $n'_{r+1} > \partial(F(\nu_i, \mu_i))$, $F(\nu_i, \mu_i) \in P(n'_1, \dots, n'_{r+1}) \cap K[x_1, \dots, x_r]$, $1 \leq i \leq m$, $|P(n'_1, \dots, n'_{r+1})| \geq m$ and $P(r+1) = \infty$ follows by [3].

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