ON THE FRATTINI SUBGROUP OF A RESIDUALLY FINITE GENERALIZED FREE PRODUCT

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ABSTRACT. Let $G = (A * B)_H$ be the generalized free product of the groups A, B amalgamating the subgroup H , and let $\Phi(G)$ denote its Frattini subgroup. In support of the conjecture that $\Phi(G) \subseteq H$ whenever G is residually finite and H satisfies a nontrivial identical relation, we show, amongst several other things, that the above inequality is indeed valid if in addition at least one of the following holds: (i) A, B , each satisfies a nontrivial identical relation; (ii) G is finitely generated; (iii) H is nilpotent. In particular (i) completes earlier investigations of the second author. The methods of proof are, however, different.

1. The questions concerning the Frattini subgroup of a generalized free product of groups, first raised by Higman and Neumann [6] have, so far, been answered only for certain special classes of groups. In [7, Theorem 3.5] the second author proved the following result (throughout we assume $A \neq H \neq B$): If $G = (A * B)_H$ is residually finite (in symbols, $G \in R\mathcal{F}$) then $\Phi(G)$ is contained in H provided one of the following conditions is satisfied: (i) A, B each satisfies a nontrivial identical relation which is not satisfied by the infinite dihedral group; (ii) A, B each satisfies a nontrivial identical relation and both $|A:H|$ and $|B:H|$ are greater than 2; (iii) $|A:H| = |B:H| = 2$.

In this paper we improve this result to read:

Theorem 1. Let $G = (A * B)_H \in R\mathcal{F}$. If $H \subsetneq C \subseteq A$ and $H \subsetneq D \subseteq B$, where C, D each satisfies a nontrivial identical relation, then $\Phi(G) \subseteq H$.

It is more convenient to begin with a proof of our main result, namely:

Theorem 2. Let $G = (A * B)_H \in R\mathcal{F}$. If H satisfies a nontrivial identical relation and if G is finitely generated then $\Phi(G) \subseteq H$.

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The proof of Theorem 2 leads naturally to

Theorem 3. *Let $G = (A * B)_H \in R\mathcal{F}$. If H satisfies a nontrivial identical relation and if A, B possess proper subgroups K, L of finite index containing H , then $\Phi(G) \subseteq H$.*

This in turn leads, on closer investigation, to a proof of Theorem 1 and also to

Theorem 4. *Let $G = (A * B)_H \in R\mathcal{F}$. If neither A nor B lies in the variety generated by H then $\Phi(G) \subseteq H$.*

Finally a result of Djokovic and Tang [3] helps us prove

Theorem 5. *Let $G = (A * B) \in R\mathcal{F}$. If H is nilpotent then $\Phi(G) \subseteq H$.*

2. In what follows we shall need the following result from [1].

Theorem A. *Let $G = (A * B)_H$. If G contains a nontrivial normal subgroup N such that $N \cap H = 1$ then $\Phi(G) \subseteq H$.*

As an important consequence of this and work of Dyer [4] we have

Theorem 0. *Let $G = (A * B)_H$. Suppose $X \triangleleft A$, $Y \triangleleft B$ are finite and such that $X \cap H = Y \cap H$ and not both of X, Y are contained in H . Then $\Phi(G) \subseteq H$.*

Proof. From the proof of Dyer's Proposition 1 [4, p. 133] we see that the normal closure $\langle X, Y \rangle^G$ of $\langle X, Y \rangle$ in G is an extension of a free group by a finite group of order n , say. Since at least one of X, Y is not in H the free group is seen to be nontrivial and consequently $\langle X, Y \rangle^G$ contains a nontrivial characteristic free subgroup (namely that generated by the n th powers of its elements). This subgroup is clearly normal in G and intersects H trivially whence $\Phi(\bar{G}) \subseteq H$ by Theorem A.

We can now give the

Proof of Theorem 2. First suppose $\Phi(G) \cap A = \Phi(G) \cap B = \Phi(G) \cap H = T$, say. Then $T \triangleleft G$. Passing to $G/T = (A/T * B/T)_{H/T}$ we see that G/T contains the subgroup $\Phi(G)/T = \Phi(G/T)$, a normal subgroup which intersects H/T trivially. It follows immediately from Theorem A that $\Phi(G/T) \subseteq H/T$ and hence that $\Phi(G) \subseteq H$.

Thus we may suppose that there exists $a \in A$ (say) such that $a \in \Phi(G)$ but $a \notin H$. From this it follows that $|A:H| > 2$: for, if $|A:H| = 2$ we may choose $1, a$ as coset representatives of A modulo H and then obtain the

obvious contradiction $G = \langle A, B \rangle = \langle a, H, B \rangle = \langle a, B \rangle = B$ since $a \in \Phi(G)$, the set of nongenerators of G .

Now select $b \in B \setminus H$ arbitrarily and let a_1 be an element of A not in $H \cup aH$. Clearly $u = a_1^{-1}(a \cdot b^{-1}ab)a_1$ is an element of $\Phi(G)$ which, in reduced form, has both its initial and final letter in $A \setminus H$, since $a_1^{-1}a \notin H$. Let $S = \langle u, b^{-1}ub \rangle$. It is easily seen that S is a free group of rank 2 and that $S \subseteq \Phi(G)$. Let $w(x_1, \dots, x_n) = 1$ be a nontrivial identical relation satisfied by H and let $y_1, \dots, y_n$ be any n elements of S which freely generate a subgroup of rank n . Then $w(y_1, \dots, y_n)$ is a nontrivial element of S .

Since $G \in \mathcal{RF}$ there exists a normal subgroup N of finite index in G such that $w(y_1, \dots, y_n) \notin N$. Setting $U = N \cap A \cap \Phi(G)$, $V = N \cap B \cap \Phi(G)$ we see that $U \triangleleft A$, $V \triangleleft B$ and $U \cap H = H \cap V$. Consequently we can form the factor amalgam $(A/U, B/V; H/U \cap V)$ and its generalized free product $\bar{G}$. Now $\bar{G}$ is a proper free product: from $HU = A$ it follows that $H(\Phi(G) \cap A) = A$ whence $G = \langle A, B \rangle = \langle H, \Phi(G) \cap A, B \rangle = \langle \Phi(G), B \rangle = B$ since G is finitely generated (see, for example, [2]), an obvious contradiction. If ψ denotes the natural map from G to $\bar{G}$ we see that $\ker \psi \subseteq N \cap \Phi(G)$. Further $a \in A \cap \Phi(G)$. However $a \notin \ker \psi$ since $w(y_1, \dots, y_n)$ and consequently a is not in N . Thus $\bar{A} = A/U$ contains the nontrivial finite normal subgroup $\bar{A} \cap \Phi(\bar{G}) \cong A \cap \Phi(G)/U$, $\bar{B} = B/V$ contains the finite normal subgroup $\bar{B} \cap \Phi(\bar{G}) \cong B \cap \Phi(G)/V$ and the intersections of these subgroups with $\bar{H}$ are equal. Thus either $\bar{A} \cap \Phi(\bar{G}) = \bar{B} \cap \Phi(\bar{G})$ in which case $\Phi(\bar{G}) \subseteq \bar{H}$ as at the beginning of this proof, or at least one of $\bar{A} \cap \Phi(\bar{G})$, $\bar{B} \cap \Phi(\bar{G})$ properly contains $\bar{H} \cap \Phi(\bar{G})$ so that Theorem 0 is applicable to show again that $\Phi(\bar{G}) \subseteq \bar{H}$.

Thus a maps under ψ onto $a\psi \in \Phi(\bar{G}) \subseteq \bar{H}$. Hence $w(y_1, \dots, y_n)\psi \in \bar{H}$: further each $y_i\psi \in \Phi(\bar{G}) \subseteq \bar{H}$, hence $w(y_1, \dots, y_n)\psi = 1$ in $\bar{H}$, i.e. $w(y_1, \dots, y_n) \in \ker \psi \subseteq N$, a contradiction. Thus Theorem 2 is proved.

Another condition which ensures $HU \neq A$ and $HV \neq B$ as required in the proof of Theorem 2 is the existence in A, B of proper subgroups of finite index which contain H . Indeed, if $H \subseteq K \subsetneq A$ and $|A:K| < \infty$, then $HU \neq A$. For otherwise, $K(\Phi(G) \cap A) \supseteq HU = A$ and consequently A is generated by K and finitely many elements $\phi_1, \dots, \phi_t$ of $\Phi(G)$. This leads to $G = \langle A, B \rangle = \langle \phi_1, \dots, \phi_t, K, B \rangle = \langle K, B \rangle = (K * B)_H \subsetneq G$, a contradiction. Thus we have proved Theorem 3.

To prove Theorem 4 we note that H satisfies a nontrivial identical re-

lation; for, if not, H generates the variety of all groups, contradicting the fact that A and B do not lie in this variety. Now $G \in R^{\mathcal{F}}$, G possesses a collection N_λ ($\lambda \in \Lambda$) of normal subgroups of finite index such that $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. Theorem 3 shows that if there exist $\mu, \nu \in \Lambda$ such that $H(A \cap N_\mu) \neq A$ and $H(B \cap N_\nu) \neq B$ (since H satisfies a nontrivial identical relation) then $\Phi(G) \subseteq H$. At the other extreme if $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ and if $H(B \cap N_\lambda) = B$ for all $\lambda \in \Lambda$, then again $\Phi(G) \subseteq H$. For, then, choosing $N = N_\lambda \triangleleft G$ and of finite index as before so that $w(y_1, \dots, y_n) \notin N$ we see that $aN \in HN/N$, $y_i N \in HN/N$ and $w(y_1, \dots, y_n)N = w(y_1 N, \dots, y_n N) = 1$ in HN/N , i.e. $w(y_1, \dots, y_n) \in N$, a contradiction. Thus if G is to be such that $\Phi(G) \not\subseteq H$ we must have either $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ or $H(B \cap N_\lambda) = B$ for all $\lambda \in \Lambda$, but not both. It then follows that either A or B must satisfy the same identical relations as H . This proves Theorem 4.

To prove Theorem 1 we suppose, as before, that there exists $a \in A \cap \Phi(G)$ such that $a \notin H$. By the remarks above we may also suppose that either (i) $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$ or (ii) $H(B \cap N_\lambda) = B$ for all $\lambda \in \Lambda$, but not both. Suppose (ii) holds. Then there exists $\mu \in \Lambda$ such that $H(A \cap N_\mu) \neq A$. Consequently $H(A \cap \Phi(G) \cap N_\mu \cap N_\nu) \neq A$ for each $\nu \in \Lambda$. Thus if $\Phi(G) \cap B = \Phi(G) \cap H$ the factor amalgam $(A/U, B/V; H/U \cap V)$, where $U = A \cap \Phi(G) \cap N_\mu \cap N_\nu$, $V = B \cap \Phi(G) \cap N_\nu \cap N_\mu$, is a proper amalgam for all $\nu \in \Lambda$ and we can proceed as before. Hence if (ii) holds and $\Phi(G) \not\subseteq H$ there must exist $b \in B \cap \Phi(G)$ such that $b \notin H$. Therefore we may assume, without loss of generality, that (i) holds. Consider the generalized free product $(A * D)_H$. It is residually finite and it follows (because of (i)) from Theorem 2 of Gregorac [5] that H has index either 1 or 2 in $\bigcap_{\lambda} H(A \cap N_\lambda)$; that is, H is of index 1 or 2 in A . Both of these possibilities are, however, prohibited; the first trivially and the second by the remarks in the second paragraph of the proof of Theorem 2. This proves Theorem 1.

Next we give the

Proof of Theorem 5. By the remarks above we may assume that (i) there exists $a \in A \setminus H$ such that $a \in \Phi(G)$; (ii) $H(A \cap N_\lambda) = A$ for all $\lambda \in \Lambda$. Then (α) A is nilpotent (of the same class as H) so that $H^A \neq A$ and (β) there exists $\mu \in \Lambda$ such that $H(B \cap N_\mu) \neq B$. As usual we now select $\nu \in \Lambda$ such that $w(y_1, \dots, y_n) \notin N_\nu$. Set $N = N_\nu \cap N_\mu$. Now look at the (obviously proper) generalized free product $\bar{G}$ on the amalgam $(A/N \cap H^A, B/N \cap B; H/H \cap N)$. Let ψ denote the natural homomorphism

from G onto $\bar{G}$. Then $a\psi \in \Phi(\bar{G})$. But $\Phi(\bar{G}) \subseteq H/H \cap N$ by Theorem 1 of [3] since $H/H \cap N$ is a finite group. As before it follows that $w(y_1, \dots, y_n) \in \ker \psi \subseteq N$ —a contradiction.

Finally we state formally the conjecture: If $G = (A * B)_H \in R^{\mathcal{F}}$ and if H satisfies a nontrivial identical relation, then $\Phi(G) \subseteq H$.

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