FREE CYCLIC ACTIONS ON $S^1 \times S^1 \times S^1$

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ABSTRACT. If the cyclic group Z_p acts freely on the 3-torus $T^3 = S^1 \times S^1 \times S^1$, then the quotient manifold T^3/Z_p is shown to be one of seven specifically described 3-manifolds, each of which is a T^2 bundle over S^1 . Furthermore the covering projection $T^3 \rightarrow T^3/Z_p$ can be factored as a standard covering $T^3 \rightarrow T^3$ followed by a 1, 2, 3, 4, or 6 sheeted covering $T^3 \rightarrow T^3/Z_p$. In the process the action of Z_p on T^3 is classified up to equivalence.

1. Introduction. There is little, in general, known about the classification of finite group actions on compact 3-manifolds. For example, the free actions of cyclic groups on the 3-sphere, S^3 , have only been classified for orders two [3], four [5], and eight [6]. For 3-manifolds with more structure, however, the work of Waldhausen [7] allows us to reduce the problem to a group-theoretic one which, in some cases, is quite tractable. We here illustrate this by classifying all free actions of cyclic groups, Z_p , on the 3-torus, $T^3 = S^1 \times S^1 \times S^1$, or what turns out to be the same thing, classifying the quotient manifolds T^3/Z_p .

We begin by describing (the only) seven 3-manifolds which are quotients of free cyclic actions on T^3 . We regard S^1 as the additive group of reals modulo 1. If h is an autohomeomorphism of $T^2 = S^1 \times S^1$, we denote by $(T^2 \times I)/h$ the torus bundle over S^1 obtained from $T^2 \times I$ by identifying each $(x, 0) \in T^2 \times \{0\}$ with $(h(x), 1) \in T^2 \times \{1\}$.

The following list describes $M_i = (T^2 \times I)/h_i$, $\pi_1(M_i)$, $H_1(M_i)$, homeomorphisms $\tau_{i,p}: T^3 \rightarrow T^3$ (p as restricted) which generate free actions of Z_p on T^3 with quotient manifold M_i , and free generators for $f_{i,p*}(\pi_1(T^3))$, where $f_{i,p}: T^3 \rightarrow M_i$ is the associated covering projection.

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- (1.1)
$$\begin{aligned} M_1 &= T^3, \quad h_1 = 1; \\ \pi_1(M_1) &= \langle a, b, c : [a, b] = [a, c] = [b, c] = 1 \rangle; \\ H_1(M_1) &= Z \oplus Z \oplus Z; \\ \tau_{1,p}(t_1, t_2, t_3) &= (t_1, t_2, t_3 + 1/p); \\ f_{1,p*}(\pi_1(T^3)) &= gp(a, b, c^p); \end{aligned}$$
- (1.2)
$$\begin{aligned} h_2(t_1, t_2) &= (-t_1, -t_2); \\ \pi_1(M_2) &= \langle a, b, c : [a, b] = 1, cac^{-1} = a^{-1}, cbc^{-1} = b^{-1} \rangle; \\ H_1(M_2) &= Z \oplus Z_2 \oplus Z_2; \\ \tau_{2,2k}(t_1, t_2, t_3) &= (-t_1, -t_2, t_3 + 1/2k); \\ f_{2,2k*}(\pi_1(T^3)) &= gp(a, b, c^{2k}); \end{aligned}$$
- (1.3)
$$\begin{aligned} h_3(t_1, t_2) &= (t_1, -t_2); \\ \pi_1(M_3) &= \langle a, b, c : [a, b] = 1, cac^{-1} = a, cbc^{-1} = b^{-1} \rangle; \\ H_1(M_3) &= Z \oplus Z \oplus Z_2; \\ \tau_{3,2k}(t_1, t_2, t_3) &= (t_1, -t_2, t_3 + 1/2k); \\ f_{3,2k*}(\pi_1(T^3)) &= gp(a, b, c^{2k}); \end{aligned}$$
- (1.4)
$$\begin{aligned} h_4(t_1, t_2) &= (t_1 + t_2, -t_2); \\ \pi_1(M_4) &= \langle a, b, c : [a, b] = 1, cac^{-1} = a, cbc^{-1} = ab^{-1} \rangle; \\ H_1(M_4) &= Z \oplus Z; \\ \tau_{4,2k}(t_1, t_2, t_3) &= (t_1 + t_2, -t_2, t_3 + 1/2k); \\ f_{4,2k*}(\pi_1(T^3)) &= gp(a, b, c^{2k}); \end{aligned}$$
- (1.5)
$$\begin{aligned} h_5(t_1, t_2) &= (-t_2, t_1 - t_2); \\ \pi_1(M_5) &= \langle a, b, c : [a, b] = 1, cac^{-1} = b, cbc^{-1} = a^{-1}b^{-1} \rangle; \\ H_1(M_5) &= Z \oplus Z_3; \\ \tau_{5,3k}(t_1, t_2, t_3) &= (-t_2, t_1 - t_2, t_3 + 1/3k); \\ f_{5,3k*}(\pi_1(T^3)) &= gp(a, b, c^{3k}); \end{aligned}$$

$$\begin{aligned}
 (1.6) \quad & h_6(t_1, t_2) = (t_2, -t_1); \\
 & \pi_1(M_6) = \langle a, b, c: [a, b] = 1, cac^{-1} = b^{-1}, cbc^{-1} = a \rangle; \\
 & H_1(M_6) = \mathbb{Z} \oplus \mathbb{Z}_2; \\
 & \tau_{6,4k}(t_1, t_2, t_3) = (t_2, -t_1, t_3 + 1/4k); \\
 & f_{6,4k*}(\pi_1(T^3)) = gp(a, b, c^{4k});
 \end{aligned}$$

$$\begin{aligned}
 (1.7) \quad & h_7(t_1, t_2) = (-t_2, t_1 + t_2); \\
 & \pi_1(M_7) = \langle a, b, c: [a, b] = 1, cac^{-1} = b, cbc^{-1} = a^{-1}b \rangle; \\
 & H_1(M_7) = \mathbb{Z}; \\
 & \tau_{7,6k}(t_1, t_2, t_3) = (-t_2, t_1 - t_2, t_3 + 1/6k); \\
 & f_{7,6k*}(\pi_1(T^3)) = gp(a, b, c^{6k}).
 \end{aligned}$$

We note that the above manifolds are distinguished by their homology groups.

2. Results. If M is a 3-manifold, we denote by $\hat{M}$ the 3-manifold obtained from M by capping off each 2-sphere in ∂M with a 3-cell.

Theorem 1. Suppose M is a compact 3-manifold which contains no fake 3-cells. If there is an exact sequence $1 \rightarrow A \rightarrow \pi_1(M) \rightarrow \mathbb{Z}_p \rightarrow 1$, where A is free abelian of rank 3, then $\hat{M}$ is (homeomorphic to) one of the manifolds M_i described in (1).

Theorem 2. If $\mathbb{Z}_p$ acts freely on T^3 , $M = T^3/\mathbb{Z}_p$, and $f: T^3 \rightarrow M$ is the covering projection, then

(i) $M = M_i$ ($1 \leq i \leq 7$)

for some M_i given in (1) and the action is equivalent to the one generated by $\tau_{i,p}$, and

(ii) there is a commutative diagram

$$\begin{array}{ccc}
 & T^3 & \\
 f \downarrow & \searrow g & \\
 & T^3 & \\
 & \swarrow f_1 & \\
 & M &
 \end{array}$$

of covering maps, where f_1 is a 1, 2, 3, 4, or 6 sheeted covering according as $i = 1$, $2 \leq i \leq 4$, $i = 5$, $i = 6$, or $i = 7$.

Proofs. We note that for M as in Theorem 2, $\pi_1(M)$ fits into an exact sequence as in Theorem 1 with $A = f_*\pi_1(T^3)$. We assume, by adding 3-cells if necessary, that $M = \hat{M}$. Thus if M satisfies the hypotheses of either Theorem 1 or Theorem 2 we have

2.1 $\pi_2(M) = 0$. This is immediate in Theorem 2. In Theorem 1, let $p: \tilde{M} \rightarrow M$ be the covering space with $p_*\pi_1(\tilde{M}) = A$. Since A is torsion free and is neither Z nor a free product, and since $\partial\tilde{M}$ contains no 2-spheres, it follows from the sphere theorem [4] and the projective plane theorem [1] that $\pi_2(\tilde{M}) (= \pi_2(M)) = 0$. We note that this forces $\tilde{M} = T^3 \# \Sigma$, where Σ is a homotopy 3-sphere. Once we show that M is irreducible and sufficiently large, it will follow from Theorem 8.1 of [7] that the universal cover of M is R^3 and, hence, $\tilde{M} = T^3$. We do not need this information now.

2.2 M is P^2 -irreducible. This follows because $\pi_2(M) = 0$ and M contains no fake 3-cells.

2.3 $\pi_1(M)$ is torsion free. Otherwise by [1, Theorem 8.2] we would have $\pi_2(M) \neq 0$.

Now, by Theorem 3 below, for some i ($1 \leq i \leq 7$), there is an isomorphism $\theta: \pi_1(M) \rightarrow \pi_1(M_i)$ with $\theta(A) = f_{i,p*}(\pi_1(T^3))$. Thus by [7, Theorem 6.1] or [2, Theorem A] there is a homeomorphism $h: M \rightarrow M_i$ such that $h_* = \theta$. In Theorem 2 we see that h is covered by a homeomorphism $\tilde{h}: T^3 \rightarrow T^3$ which establishes the equivalence of actions. The factorization of f promised by the second conclusion of Theorem 2 follows easily for $f = f_{i,p}$ and, hence, for any f by the equivalence.

3. Some extensions of $Z \oplus Z \oplus Z$ by Z_p .

Theorem 3. *Suppose G is a torsion free group. If there is an exact sequence*

$$(3.1) \quad 1 \rightarrow A \rightarrow G \xrightarrow{\eta} Z_p \rightarrow 1$$

where A is free abelian of rank 3, then there is an isomorphism $\theta: G \rightarrow \pi_1(M_i)$ ($1 \leq i \leq 7$) for some M_i given in (1) such that $\theta(A) = f_{i,p*}(\pi_1(T^3))$.

Proof. We fix a generator w for Z_p . We have an automorphism $\phi: A \rightarrow A$ determined as follows. Choose $z \in G$ such that $\eta(z) = w$; then $\phi(a) = zaz^{-1}$ for $a \in A$. Since A is abelian, ϕ is independent of the choice of z . Note that $\phi^p = 1$ and for $a \in A$, and any k , $z^k a = \phi^k(a)z^k$. Now $z^p \in A$, and $z^p = zz^p z^{-1} = \phi(z^p)$. Since G is torsion free, $z^p \neq 1$. Further, since $\phi(a^n) = a^n$ ($n \neq 0$) implies $\phi(a) = a$, there is a rank 1 direct summand of A which is left pointwise fixed by ϕ . So we have

3.2 There is a free basis y_1, y_2, y_3 for A such that $\phi(y_1) = y_1$ (hence $[y_1, z] = 1$), and such that $z^p = y_1^n$ for some $n \neq 0$. It follows that $(p, n) = 1$; for if $(p, n) = d > 1$, then $(y_1^{n/d} z^{-p/d})^d = 1$, while $y_1^{n/d} z^{-p/d} \neq 1$, contrary to the assumption that G be torsion free.

Now suppose that ϕ has order p_1 . Thus $p = p_1 k$ for some k . Then $A_1 = \langle y_1, y_2, y_3, z^{p_1} \rangle$ is abelian and normal and $G/A_1 \cong Z_{p_1}$. In fact A_1 is freely generated by $x_1 = y_1^s z^{r p_1}$ (where $m + sk = 1$), $x_2 = y_2$ and $x_3 = y_3$. Furthermore $y_1 = x_1^k, z^{p_1} = x_1^n$.

The action of z on A extends, by the identity on $gp(x_1)$, to an action on A_1 which we still denote by ϕ . Now G is completely determined by the action ϕ and the element $z^{p_1} \in A_1$. In particular G has a presentation

$$(3.3) \quad \begin{aligned} G = \langle x_1, x_2, x_3, z : [x_i, x_j] = 1, zx_1z^{-1} = x_1, \\ zx_2z^{-1} = w(\phi(x_2)), zx_3z^{-1} = w(\phi(x_3)), z^{p_1} = x_1^n \rangle, \end{aligned}$$

where $w(\phi(x_i))$ is any word in x_1, x_2, x_3 representing the element $\phi(x_i) \in A_1$. We further note that ϕ has order p_1 , $(p_1, n) = 1$, we may assume (by replacing z by $x_1^m z^{\pm 1}$) that $1 \leq n \leq [p_1/2]$, and A is the subgroup generated by $\{x_1^k, x_2, x_3\}$.

To analyze the automorphism ϕ , we note that, since $gp(x_1)$ is invariant under ϕ , we have an induced automorphism on the quotient group. This is described by:

3.4 Lemma. If ψ is a periodic automorphism of $B = Z \oplus Z$ (written additively), then there is a basis v_1, v_2 for B such that the matrix $Q = (q_{ij})$ of ψ ($\psi(v_i) = q_{i1}v_1 + q_{i2}v_2$) is one of

$$\begin{aligned} \text{(i)} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{(ii)} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{(iii)} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \text{(iv)} \quad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \text{(v)} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \text{(vi)} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \text{(vii)} \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

In particular ψ has order 1, 2, 3, 4, or 6.

Proof. If $\psi = \pm 1$ we have (i) or (ii). Otherwise the minimum polynomial of ψ is a quadratic polynomial with integer coefficients and with distinct

p th roots of unity ($p = \text{order } \psi$) as roots and must therefore be one of $x^2 - 1$, $x^2 + 1$, $x^2 - x + 1$, $x^2 + x + 1$.

The proof can be completed by straightforward calculations. We illustrate one case. Suppose the minimum polynomial of ψ is $x^2 - x + 1$. Of all bases for B , choose one which minimizes $|q_{11}|$, note that $q_{11} + q_{22} = 1$ and $q_{11}q_{22} - q_{12}q_{21} = 1$. It follows that $q_{11} = 0$. If not, then one of $|q_{12}| \leq |q_{11}|$, $|q_{21}| \leq |q_{11}|$ must hold; otherwise

$$q_{11}^2 - q_{11} + 1 = |q_{11}^2 - q_{11} + 1| = |q_{12}q_{21}| \geq (|q_{11}| + 1)^2$$

which is impossible if $q_{11} \neq 0$. Since conjugation of Q by $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$ (or by $\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$) replaces q_{11} by $q_{11} \pm q_{21}$ (or by $q_{11} \pm q_{12}$), we can reduce $|q_{11}|$. Thus $q_{11} = 0$; hence $q_{22} = 1$ and $q_{12}q_{21} = -1$. Thus we either have (vii) or (in case $q_{12} = -1$, $q_{21} = 1$) we reduce to (vii) by replacing v_1 by $-v_1$.

To complete the proof of Theorem 3, we note that with respect to the basis x_1, x_2, x_3 for A (which we now write additively) the matrix of ϕ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & Q \\ \beta & \end{pmatrix},$$

where Q is one of the matrices given in Lemma 3.4.

Corresponding to the possibilities for Q we have the following cases which we analyze using (3.3).

(i) Periodicity forces $\alpha = \beta = 0$. Thus $p_1 = 1$. The isomorphism $\theta: G \rightarrow \pi_1(M_1)$ is given by $\theta(x_1) = c$, $\theta(x_2) = a$, $\theta(x_3) = b$.

(ii) By replacing x_2 by $x'_2 = -\alpha x_1/2 + x_2$ (α even), or by $x'_2 = (1 - \alpha)x_1/2 + x_2$ (α odd), we may assume $\alpha = 0, 1$. Similarly we may assume $\beta = 0, 1$. By interchanging x'_2 and x'_3 if necessary, or by replacing x'_2 by $x_1 - x'_2 - x'_3$ if $\alpha = \beta = 1$, we may further assume that $\alpha = 0$ and $\beta = 0, 1$. We note that $p_1 = 2$ and, hence, $n = 1$. If $\beta = 0$ we get the isomorphism $\theta: G \rightarrow \pi_1(M_2)$ by $\theta(x_1) = c^2$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(z) = c$. If $\beta = 1$, one checks that $(x_3 z^{-1})^2 = 1$, contrary to the assumption that G be torsion free.

(iii) Periodicity forces $\alpha = 0$ and, as above, we may assume $\beta = 0, 1$. So $p_1 = 2$ and $n = 1$. If $\beta = 0$ we get the isomorphism $\theta: G \rightarrow \pi_1(M_3)$ by $\theta(x_1) = c^2$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(z) = c$. If $\beta = 1$, one has $(x_3 z^{-1})^2 = 1$, contrary to assumption.

(iv) We must have $\alpha = 0$, and may assume $\beta = 0, 1$. If $\beta = 1$ we

replace x_2 by $x_1 + x_2$ and see that we may assume $\alpha = \beta = 0$. So $p_1 = 2$, $n = 1$, and we get an isomorphism $\theta: G \rightarrow \pi_1(M_4)$ by $\theta(x_1) = c^2$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(z) = c$.

(v) By replacing x_2 by $\gamma x_1 + x_2$ and x_3 by $(\gamma + \alpha)x_1 + x_3$ for appropriate γ , we may assume $\alpha = 0$ and $\beta = 0, 1, 2$. Thus $p_1 = 3$ and $n = 1$. If $\beta = 0$, we get an isomorphism $\theta: G \rightarrow \pi_1(M_5)$ by $\theta(x_1) = c^3$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(z) = c$. If $\beta = 1$, then $(x_2 z^{-1})^3 = 1$, while if $\beta = 2$, $(x_2^{-1} z^2)^3 = 1$. In either case we have a contradiction.

(vi) By replacing x_2 by $(\beta - \alpha)x_1/2 + x_2$ and x_3 by $-(\beta + \alpha)x_1/2 + x_3$ if $\alpha + \beta$ is even, or by replacing x_2 by $(\beta - \alpha - 1)x_1/2 + x_2$ and x_3 by $(1 - \alpha - \beta)x_1/2 + x_3$ if $\alpha + \beta$ is odd, we may assume $\alpha = 0$, $\beta = 0, 1$. Now $p_1 = 4$, $n = 1$; so if $\beta = 0$ we get an isomorphism $\theta: G \rightarrow \pi_1(M_6)$ by $\theta(x_1) = c^4$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(z) = c$. If $\beta = 1$, we have the contradiction $(z^{-2} x_2^{-1})^2 = 1$.

(vii) By replacing x_2 by $-\beta x_1 + x_2$ and x_3 by $(\alpha - \beta)x_1 + x_3$, we may assume $\alpha = \beta = 0$. Now $p_1 = 6$, $n = 1$; so we get an isomorphism $\theta: G \rightarrow \pi_1(M_7)$ by $\theta(x_1) = c^6$, $\theta(x_2) = a$, $\theta(x_3) = b$, $\theta(z) = c$.

We conclude by noting that in each case $\theta(A) = f_{i,p*}(\pi_1(T^3))$.

REFERENCES

1. D. B. A. Epstein, *Projective planes in 3-manifolds*, Proc. London Math. Soc. (3) **11** (1961), 469–484. MR **27** #2968.
2. Wolfgang Heil, *On P^2 -irreducible 3-manifolds*, Bull. Amer. Math. Soc. **75** (1969), 772–775. MR **40** #4958.
3. G. R. Livesay, *Fixed point free involutions on the 3-sphere*, Ann. of Math. (2) **72** (1960), 603–611. MR **22** #7131.
4. Ch. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*, Ann. of Math. (2) **66** (1957), 1–26. MR **19**, 761.
5. P. M. Rice, *Free Z_8 actions on S^3* , Duke Math. J. **36** (1969), 749–751. MR **40** #2064.
6. Gerhard X. Ritter, *Free Z_8 actions on S^3* , Trans. Amer. Math. Soc. **18** (1973), 195–212.
7. Friedhelm Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR **36** #7146.

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