THE DOMINION AND SEPARABLE SUBALGEBRAS OF FINITELY GENERATED ALGEBRAS¹

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ABSTRACT. Let $R \subseteq S \subseteq A$ be rings with R commutative, A an R -algebra which is finitely generated as an R -module by n -elements, and S a subalgebra of A . Setting $D_0 = S$ and $D_i = \text{Dom}(R, D_{i-1})$ for $i \geq 1$, we show that $\bigcap_n D_n = R$. We use this to show that if S is a separable R -algebra then S is also finitely generated as an R -module. Finally, a characterization of S is given when A is commutative and S is an inertial subalgebra of A .

1. Preliminaries. Throughout this paper all rings have identity, subrings contain the identity of the larger ring, and ring homomorphisms map the identity to the identity. With the exception of Definition 1 and Theorem 1, R denotes a commutative ring and S denotes an R -algebra; that is, S is a ring together with a ring homomorphism f from R into the center of S . It will be no real restriction to assume that S is faithful as an R -module under the module structure induced by f ; thus, we may consider R as a subring of the center of S .

In any category a morphism f is called an *epimorphism* if for any two morphisms g and h the equality of $g \circ f$ and $h \circ f$ implies the equality of g and h . A ring S will be called an *epimorphic extension* of a commutative ring R if S is a faithful R -algebra and the natural injection of R into S is an epimorphism.

2. The dominion of R in S .

Definition 1. Let R and S be rings, not necessarily commutative, and let $f: R \rightarrow S$ be a ring homomorphism. The *f -dominion of R in S* , denoted

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$\text{Dom}_f(R, S)$, is the set of elements s of S such that: If g and h are any two ring homomorphisms from S to a ring T with $g \circ f = h \circ f$, then $g(s) = h(s)$ [5, p. 382].

It is easily seen that $\text{Dom}_f(R, S)$ is a subring of S containing $f(R)$. In addition $f: R \rightarrow S$ is an epimorphism if and only if $\text{Dom}_f(R, S) = S$. Whenever R is a subring of S and f is the natural injection of R into S , we write $\text{Dom}(R, S)$.

The following theorem gives two of the many characterizations of $\text{Dom}_f(R, S)$.

Theorem 1. *Let R and S be rings, not necessarily commutative, and let $f: R \rightarrow S$ be a ring homomorphism. The following statements are equivalent:*

- (1) $s \in \text{Dom}_f(R, S)$.
- (2) The relation $s \otimes 1 - 1 \otimes s = 0$ holds in $S \otimes_R S$.
- (3) There exist matrices σ_1 and σ_2 with entries from S and a matrix θ with entries from $f(R)$ such that:
 - (i) $(s) = \sigma_1 \theta \sigma_2$ and
 - (ii) both $\sigma_1 \theta$ and $\theta \sigma_2$ have entries from $f(R)$.

Proof. The equivalence of (1) and (2) has been shown by Roby [4, Theorem 1]. The equivalence of (1) and (3) has been shown by Mazet [3, Theorem 1].

Due to the nature of the computations in the remainder of this section, some preliminaries are in order.

Let S be a ring and M a left S -module. Let $\mathcal{S}$ and $\mathcal{M}$ denote the set of matrices having entries from S and M respectively. Define addition in $\mathcal{M}$ and $\mathcal{S}$ and multiplication in $\mathcal{S}$ as usual. If $\sigma = (s_{ij}) \in \mathcal{S}$ is $m \times n$ and $\mu = (m_{ij}) \in \mathcal{M}$ is $n \times p$ we define, quite naturally, $\sigma\mu$ to be the $m \times p$ matrix whose i, j entry is $\sum_{k=1}^n s_{ik} m_{kj}$. If μ is $m \times n$ and σ is $n \times p$ we define $\mu \cdot \sigma = (\sigma' \mu')^t$ where $()^t$ denotes the transpose matrix. If $x \in M$ we define σx to be the matrix $(s_{ij}x)$ (resp. $x\sigma = (xs_{ij})$ whenever M is a right S -module).

Suppose σ, σ' are in $\mathcal{S}$, μ, μ' are in $\mathcal{M}$, and x is in M . Whenever these matrices have compatible dimensions they satisfy the following:

- (1) $\sigma\mu, \mu \cdot \sigma$ and σx are all in $\mathcal{M}$,
- (2) $(\sigma + \sigma')\mu = \sigma\mu + \sigma'\mu$, (2') $\mu \cdot (\sigma + \sigma') = \mu \cdot \sigma + \mu \cdot \sigma'$,
- (3) $\sigma(\mu + \mu') = \sigma\mu + \sigma\mu'$, (3') $(\mu + \mu') \cdot \sigma = \mu \cdot \sigma + \mu' \cdot \sigma$,
- (4) $(\sigma\sigma')\mu = \sigma(\sigma'\mu)$,
- (5) $(\sigma\sigma')x = \sigma(\sigma'x)$.

If S is commutative, then

$$(6) \mu \cdot (\sigma\sigma') = (\mu \cdot \sigma) \cdot \sigma',$$

$$(7) \sigma(\sigma'x) = (\sigma x) \cdot \sigma'.$$

Properties 2', 3' and 1–5 are immediate. Properties 6 and 7 follow from the identity $(\sigma\sigma')^t = \sigma'^t\sigma^t$, which holds whenever S is commutative.

We are now ready to proceed.

Proposition 2. *Let R be a commutative ring, S a faithful R -algebra, and M a two-sided S -module. If $m \in M$ is such that $rm = mr$ for every $r \in R$, then $sm = ms$ for every $s \in \text{Dom}(R, S)$.*

Proof. Suppose $m \in M$ is such that $rm = mr$ for every $r \in R$. If $s \in \text{Dom}(R, S)$ and if σ_1, σ_2 and θ are the matrices of Theorem 1, then we have

$$\begin{aligned} (s)m &= (\sigma_1\theta\sigma_2)m = \sigma_1[(\theta\sigma_2)m] = \sigma_1[m(\theta\sigma_2)] \\ &= \sigma_1[(m\theta)\sigma_2] = \sigma_1[(\theta m)\sigma_2] = [(\sigma_1\theta)m]\sigma_2 \\ &= [m(\sigma_1\theta)]\sigma_2 = m(\sigma_1\theta\sigma_2) = m(s). \end{aligned}$$

Therefore $sm = ms$.

Corollary 2.1. *If R is commutative and S is a faithful R -algebra, then $\text{Dom}(R, S)$ is contained in the center of S .*

Proof. Set $M = S$ and note that R is in the center of S .

Corollary 2.2. *Any epimorphic extension of a commutative ring is itself commutative.*

Next we will examine finitely generated S -modules.

Lemma 3. *Let S be an R -algebra and let M be a left S -module. If a left R -submodule N of M and an element x of M are such that $M = N + Rx$, then N is a left $\text{Dom}(R, S)$ -submodule of M .*

Proof. Let s be in $\text{Dom}(R, S)$ and let n be in N ; it suffices to show that $sn \in N$. Let σ_1, σ_2 and θ be the matrices of Theorem 1. Since $M = N + Rx$ we can set $\sigma_1x = \mu_1 + \phi_1x$ and $\sigma_2n = \mu_2 + \phi_2x$, where μ_1 and μ_2 are matrices with entries from N and ϕ_1 and ϕ_2 are matrices with entries from R .

Having established notation we proceed.

$$\begin{aligned}
(sn) &= \sigma_1 \theta(\sigma_2 n) = (\sigma_1 \theta) \mu_2 + \sigma_1 [(\theta \phi_2) x] \\
&= (\sigma_1 \theta) \mu_2 + (\sigma_1 x) \cdot (\theta \phi_2) \\
&= (\sigma_1 \theta) \mu_2 + \mu_1 \cdot (\theta \phi_2) + (\phi_1 x) \cdot (\theta \phi_2) \\
&= (\sigma_1 \theta) \mu_2 + \mu_1 \cdot (\theta \phi_2) + \phi_1 \theta(\phi_2 x) \\
&= (\sigma_1 \theta) \mu_2 + \mu_1 \cdot (\theta \phi_2) + \phi_1 \theta(\sigma_2 n) - (\phi_1 \theta) \mu_2 \\
&= (\sigma_1 \theta) \mu_2 + \mu_1 \cdot (\theta \phi_2) + \phi_1 (\theta \sigma_2) n - (\phi_1 \theta) \mu_2.
\end{aligned}$$

Since N is a left R -module, the last equality expresses (sn) as a sum of four 1×1 matrices each having its entry from N . Therefore $sn \in N$.

Theorem 4. *Let S be an R -algebra and let M be a left S -module which is finitely generated as a left R -module by n elements. Set $D_0 = S$ and $D_i = \text{Dom}(R, D_{i-1})$, $i \geq 1$. Then for any element x of M , $D_n x = Rx$.*

Proof. Suppose M is generated as a left R -module by $\{x_1, \dots, x_n\}$. Set $x = x_0$ and $M_i = \sum_{j=0}^{n-i} Rx_j$, $0 \leq i \leq n$. Using Lemma 3 and a straightforward induction on i , we see that M_i is a left D_i -module for each i . In particular, $M_n = Rx$ is a left D_n -module. Therefore $D_n x \subseteq M_n = Rx$ since $x \in Rx$. Since $R \subseteq D_i$ for every i , it follows that $Rx \subseteq D_n x$ and we are done.

Corollary 4.1. *Let $R \subseteq S \subseteq A$ be rings with R -commutative, A an R -algebra which is finitely generated as an R -module by n elements, and S a subalgebra of A . Then $D_n = R$.*

Proof. Set $M = A$ and $x = 1$ in Theorem 4.

The next corollary is a strengthening of the well-known result that if an epimorphic extension S of R is finitely generated as an R -module, then $S = R$ [4, Theorem 2].

Corollary 4.2. *Let S be an epimorphic extension of R and let A be a ring containing S . If A is finitely generated as a left R -module, then $S = R$.*

Proof. Let $M = A$ and $x = 1$ in Theorem 4 and note that $D_i = S$ for every i .

3. Separable algebras. An R -algebra S is R -separable if and only if $S \bullet_R S^0$ contains an idempotent e such that $(s \otimes 1 - 1 \otimes s)e = 0$ for every s in S and $\mu(e) = 1$, where μ is the multiplication map from $S \bullet_R S^0$ to S .

defined by $\mu(\sum s_i \otimes s'_i) = \sum s_i s'_i$. The idempotent e is called a *separability idempotent* for S over R and is unique whenever S is commutative [1, p. 40].

Life would be easier if separable algebras were finitely generated as modules; however, such is not the case. For example, let R be a commutative ring and let S be any (faithful) ring of quotients of R . For each element r/w of S one has the following relations in $S \otimes_R S$:

$$\frac{r}{w} \otimes 1 = \frac{r}{w} \otimes w \left(\frac{1}{w} \right) = \frac{rw}{w} \otimes \frac{1}{w} = \frac{r}{1} \otimes \frac{1}{w} = 1 \otimes \frac{r}{w}.$$

Therefore, S is an epimorphic, a fortiori separable, R -algebra. However, S cannot be finitely generated as an R -module unless $S = R$. The next theorem, which is the main result of this paper, offers a condition under which a separable algebra is finitely generated. The proof makes use of the following results:

If S is a separable R -algebra with center C , then S is finitely generated and projective as a C -module [1, Theorem 3.4]. In addition, S is separable as a C -algebra and C is separable as an R -algebra [1, Theorem 3.8].

We are now ready for the main result of this paper.

Theorem 5. *Let R be a commutative ring and let A be an R -algebra which is finitely generated as an R -module. Then any R -separable subalgebra of A is also finitely generated as an R -module.*

Proof. Let S be an R -separable subalgebra of A and let C denote the center of S . Since S is finitely generated as a C -module, it suffices to prove that C is finitely generated as an R -module. Since C is commutative and is separable as an R -algebra, it has a unique separability idempotent $e = \sum_{i=1}^n x_i \otimes y_i$ in $C \otimes_R C$. Let R' be the R -algebra generated by $\{x_1, \dots, x_n\}$. Then R' is a subring of C . Since A is finitely generated as an R -module, each element of A is integral over R ; therefore, R' is finitely generated as an R -module [6, pp. 254–255]. We will use Corollary 4.2 to show that $R' = C$.

Since C is separable as an R -algebra, it is separable as an R' -algebra. It is easy to see that $e' = \sum_{i=1}^n x_i \otimes y_i$ in $C \otimes_{R'} C$ is the unique separability idempotent for C over R' . However, each x_i is in R' so that $e' = 1 \otimes \sum_{i=1}^n x_i y_i = 1 \otimes \mu(e) = 1 \otimes 1$ in $C \otimes_{R'} C$. It is shown on p. 40 of [1] that $C \otimes_{R'} C = (C \otimes_{R'} C) e' \otimes (C \otimes_{R'} C) (1 - e')$ where $(C \otimes_{R'} C) (1 - e')$ is the ideal in $C \otimes_{R'} C$ generated by elements of the form $s \otimes 1 - 1 \otimes s$. Since $1 - e' = 0$ one concludes $s \otimes 1 - 1 \otimes s = 0$ for all $s \in C$ so by Theorem

1(2) C is an epimorphic image of R' with $R' \subseteq C \subseteq A$ and A finitely generated as a left R' -module. Applying Lemma 4.2 we have $C = R'$ which is finitely generated as an R -module. Hence S is finitely generated as an R -module.

We conclude this paper with a uniqueness theorem for inertial subalgebras.

Let A be a finitely generated R -algebra with Jacobson radical N . A subalgebra S of A is called an *inertial subalgebra* if S is a separable R -algebra such that $A = S + N$ [2, Definition 2.1].

Lemma 6. *Let A be a finitely generated R -algebra with radical N . If B and S are subalgebras with $B \subseteq S$, S separable over R , and $B + N = A$, then $B = S$.*

Proof. Clearly S is an inertial subalgebra of A so that $\text{rad } S = S \cap N$ [2, Lemma 1.1]. By Theorem 5, S is finitely generated as an R -module; thus, $\text{rad } S = \bigcap (\mathfrak{m}S)$ where $\mathfrak{m}$ runs over all maximal ideals of R [2, Lemma 1.1]. Therefore, $S = A \cap S = (B + N) \cap S = B + (N \cap S) = B + \bigcap (\mathfrak{m}S)$, which implies $S = B + S$ for every maximal ideal $\mathfrak{m}$ of R . It follows from the generalized Nakayama lemma [6, Lemma 2, p. 215] that $S = B$.

Proposition 7. *Let A be a commutative, finitely generated R -algebra with radical N . If S is an inertial subalgebra of A , then $S = \bigcap \{B \mid B \text{ is a subalgebra of } A \text{ with } A = B + N\}$.*

Proof. Since $A = S + N$ the intersection is contained in S . Let B be a subalgebra of A such that $A = B + N$. The subalgebra BS is a ring homomorphic image of $B \otimes_R S$ and so is a separable B -algebra containing B . Applying Lemma 6 with $R = B$, we have $B = BS$ which contains S . Therefore, $S \subseteq \bigcap \{B \mid B \text{ is a subalgebra of } A \text{ with } A = B + N\}$.

Corollary 7.1. *Let A be a commutative, finitely generated R -algebra. An inertial subalgebra of A , if it exists, is unique.*

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