PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 48, Number 1, March 1975

THE PRODUCT FORMULA FOR STIEFEL-WHITNEY HOMOLOGY CLASSES

STEPHEN HALPERIN AND DOMINGO TOLEDO[1]

ABSTRACT. We give a combinatorial proof of the formula for the Stiefel-Whitney homology classes of the product of two Euler spaces. Some relevant facts on ordered triangulations are also included.

Let X be a locally finite, n-dimensional polyhedron. X is called an *integral Euler space* (*resp.* mod 2 *Euler space*) if for all $x \in X$ the local Euler characteristic $\chi(X, X - x) = (-1)^n$ (resp. $\equiv 1 \pmod 2$). If K is a triangulation of X (always assumed compatible with the PL structure of X), we denote its first barycentric subdivision by K'. If a is a vertex of K', $|a|$ is the dimension of the corresponding simplex of K. Note that the vertices of K' are naturally ordered by the inclusion of simplices in K.

For $p = 0, 1, \cdots, n$, the p'th *Stiefel chain* of K' is the chain (infinite if X is not compact)

$$(1) \qquad s_p(K') = \sum_{a_0 < \cdots < a_p} (-1)^{|a_0| + \cdots + |a_p|} \langle a_0 \cdots a_p \rangle \in C_p(K', \mathbf{Z}).$$

This is just the sum of all p-simplices of K', with appropriate signs. $s_0(K')$ is an integral cycle whose homology class represents $\chi(X)$ if X is compact and connected.

X is a mod 2 Euler space if and only if all the Stiefel chains are mod 2 cycles [3]. The homology class of $s_p(K')$ is then independent of the triangulation K (cf. [1]) and is called the p'th *Stiefel-Whitney homology class* of X, $w_p(X) \in H_p(X, \mathbf{Z}_2)$.

X is an integral Euler space if and only if $\partial s_p(K') = (1 + (-1)^{n-p}) s_{p-1}(K')$ [2]. In this case we get integral classes $w_p(X) \in H_p(X, \mathbf{Z})$ when $n - p$ is odd, and $w_p(X)$ is the Bockstein of $w_{p+1}(X) \in H_{p+1}(X, \mathbf{Z}_2)$.

If X is a smooth manifold, Whitney showed that $w_p(X)$ is the Poin-

---

Received by the editors August 8, 1973.

AMS (MOS) subject classifications (1970). Primary 57C99.

[1] Supported in part by National Science Foundation grant GP-36418 and NRC A 8047.

caré dual of the usual Stiefel-Whitney cohomology class $w^{n-p}(X)$ [4].

Now if X and Y are integral (or mod 2) Euler spaces, then so is $X \times Y$. The main purpose of this note is to give a combinatorial proof of the product formula for these classes. Here $\times$: $H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$ denotes the cross-product in homology:

Theorem. *Let* X, Y *be* mod 2 *Euler spaces. Then*

$$w_p(X \times Y) = \sum_{r=0}^{p} w_r(X) \times w_{p-r}(Y).$$

We prove this formula in §1, assuming some facts on ordered triangulations that are proved in §2. With these facts we also get as an easy corollary the product formula for the integral classes:

Corollary. *Let* K *and* L *be triangulations of integral Euler spaces* X, Y *of dimension* m, n. *Then, if* $m+n-p$ *is odd,* $\Sigma_{r=0}^{p}(-1)^{mr}s_r(K') \times s_{p-r}(L')$ *is an integral cycle that represents* $w_p(X \times Y) \in H_p(X \times Y, \mathbf{Z})$.

J. Milnor has a different combinatorial proof of the product formula. There is also an argument due to C. McCrory and D. Sullivan that reduces it to the well-known formula for manifolds via "resolutions" of Euler spaces.

For other properties of the Stiefel-Whitney homology classes see [1].

1. **Proof of the Theorem.** Let K and L be triangulations of X and Y, and let $K \times L$ denote the *cell* complex whose cells are the product of a simplex in K with a simplex in L. The vertices of its first barycentric subdivision $(K \times L)'$ are the pairs (a, b) where a is a vertex of K' and b is a vertex of L', and are ordered by $(a, b) < (a', b')$ if $a \le a'$, $b \le b'$ and $(a, b) \ne (a', b')$. The simplices of $(K \times L)'$ are precisely the linearly ordered subsets, which we always write in increasing order.

Now the cross-product in homology is induced by the chain map ($\mathbf{Z}_2$ coefficients!) $C_*(K') \otimes C_*(L') \to C_*((K \times L)')$ given by

$$(2) \quad \langle a_0 \cdots a_p \rangle \otimes \langle b_0 \cdots b_q \rangle \mapsto \sum \langle (a_{i_0}, b_{j_0}) \cdots (a_{i_{p+q}}, b_{j_{p+q}}) \rangle$$

where the sum is over all pairs $i_0 \le \cdots \le i_{p+q}$, $j_0 \le \cdots \le j_{p+q}$ such that $\{i_0, \cdots, i_{p+q}\} = \{0, \cdots, p\}$, $\{j_0, \cdots, j_{p+q}\} = \{0, \cdots, q\}$, and for each r, either $i_r < i_{r+1}$ or $j_r < j_{r+1}$.

Let $\sigma = \langle (a_0, b_0) \cdots (a_p, b_p) \rangle < (K \times L)'$. We say that σ *has a jump at* i if $a_{i-1} < a_i$ *and* $b_{i-1} < b_i$. Then clearly $\Sigma_r w_r(X) \times w_{p-r}(Y)$ is represented by the sum of all $\sigma^p < (K \times L)'$ such that σ^p has no jumps. Now by the Proposition below, $w_p(X \times Y)$ is represented by $s_p((K \times L)') =$ sum of all $\sigma^p < (K \times L)'$. Thus if

$$c_p = s_p((K \times L)') - \sum_r s_r(K') \times s_{p-r}(L'),$$

then $c_p =$ sum of all $\sigma^p < (K \times L)'$ such that σ^p has at least one jump, and the product formula is equivalent to the fact that c_p is a boundary.

We construct an explicit chain d_{p+1} such that $\partial d_{p+1} = c_p$. First we say that (a_i, b_i) is a *critical vertex* of $\tau = \langle (a_0, b_0) \cdots (a_{p+1}, b_{p+1}) \rangle$ if either

(i) $a_{i-1} < a_i = a_{i+1}$ and $b_{i-1} = b_i < b_{i+1}$, or

(ii) $a_{i-1} = a_i < a_{i+1}$ and $b_{i-1} < b_i = b_{i+1}$.

We then define an integer (mod 2), $\nu(\tau)$, by $\nu(\tau) =$ number of critical vertices of τ of type (i) before the first jump. (If τ has no jumps, $\nu(\tau)$ is the number of critical vertices of type (i).) Let

$$d_{p+1} = \sum_{\tau^{p+1} < (K \times L)'} \nu(\tau^{p+1}) \tau^{p+1}.$$

Claim. $\partial d_{p+1} = c_p$.

Proof. $\partial d_{p+1} = \Sigma_\sigma \lambda_\sigma \sigma$ where

$$\lambda_\sigma = \sum_{\tau^{p+1} > \sigma} \nu(\tau^{p+1}).$$

For each $\sigma = \langle (a_0, b_0) \cdots (a_p, b_p) \rangle$, the $(p + 1)$-simplices containing it are of the form

$$\sigma_{i;a,b} = \langle (a_0, b_0) \cdots (a_{i-1}, b_{i-1})(a, b)(a_i, b_i) \cdots (a_p, b_p) \rangle$$

(with the obvious definition of $\sigma_{0;a,b}, \sigma_{p+1;a,b}$). Then $\lambda_\sigma = \Sigma_{i=0}^{p+1} n_i(\sigma)$ where

$$n_i(\sigma) = \sum_{(a_{i-1}, b_{i-1}) < (a, b) < (a_i, b_i)} \nu(\sigma_{i;a,b}).$$

In the computations that follow, we constantly use the fact that if $x < y$ are vertices of K' or L', then each of the conditions $z < x$, $x < z < y$, $y < z$ is satisfied by an even number of vertices z. (Cf. Lemmas 1 and 2 below.)

Let $k \geq 1$ be the first jump of σ. (Possibly $k = p + 1$, i.e., σ has no jumps.) First we show that $\lambda_\sigma = n_k(\sigma)$, i.e., $n_i(\sigma) = 0$ for $i \neq k$.

Consider the following cases:

(1) $i > k$. Then $\nu(\sigma_{i;a,b}) = \nu(\sigma)$ for all (a, b), and the number of $\sigma_{i;a,b}$ is even. Thus $n_i = 0$.

(2) $i = 0$. Divide the $\sigma_{0;a,b}$ onto three disjoint subsets according to the following conditions: $a < a_0$, $b < b_0$; $a = a_0$, $b < b_0$; $a < a_0$, $b = b_0$. In each subset, $\nu(\sigma_{0;a,b})$ is constant, and the number of $\sigma_{0;a,b}$ in each subset is even. Hence $n_0 = 0$.

(3) $0 < i < k$. In this case either $a_{i-1} = a_i$ and $b_{i-1} < b_i$ or $a_{i-1} < a_i$ and $b_{i-1} = b_i$. If the first condition holds, then for all $\sigma_{i;a,b}$, $a_{i-1} = a = a_i$ and $b_{i-1} < b < b_i$. Thus $\nu(\sigma_{i;a,b})$ depends only on i (and similarly if the second condition holds). Hence, as above, $n_i = 0$.

Since $\lambda_\sigma = n_k$, the claim will be proved by showing that $n_k = 1$ if σ has jumps and 0 if σ has no jumps, i.e., it has to be shown that $n_k = 1$ if $k \leq p$ and $n_k = 0$ if $k = p + 1$.

Suppose $k = p + 1$. Then $\nu(\sigma_{p+1;a,b})$ is constant on each of the three subsets $a_{p+1} < a$, $b_{p+1} < b$; $a_{p+1} = a$, $b_{p+1} < b$; $a_{p+1} < a$, $b_{p+1} = b$; and there is an even number of vertices in each subset. Hence $n_k = 0$ in this case.

Suppose $k \leq p$. Then $\nu(\sigma_{k;a_k,b_{k-1}}) = \nu(\sigma_{k;a_{k-1},b_k}) + 1$, and ν is again constant on each of the subsets (a, b_{k-1}); (a_{k-1}, b); (a, b) with $a_{k-1} < a < a_k$, $b_{k-1} < b < b_k$. Hence in this case $n_k = 1$.

The proof of the Claim is now complete.

2. **Ordered triangulations.** By an *ordered triangulation* of a polyhedron X, we mean a triangulation K of X together with a partial ordering on the vertices such that the simplices of K are precisely the linearly ordered subsets (e.g., the first barycentric subdivision of a cell complex). We always write the simplices of K in increasing order.

If a is a vertex of K, we write simply a^- for $\langle a\rangle_0 = \{b\colon b < a\}$, and a^+ for $\langle a\rangle_1 = \{b\colon a < b\}$. Let $|a| = 1 + \dim a^-$. If K is a finite complex, $\#(K)$ means the number of vertices of K.

Note that if K is an ordered triangulation, then we can define the Stiefel chains $s_p(K) \in C_p(K, \mathbf{Z})$ by the same formula (1).

If $\sigma = \langle a_0 \cdots a_p\rangle < K$, then $Lk(\sigma, K) = \sigma_0 * \sigma_1 * \cdots * \sigma_{p+1}$, where σ_i is the full subcomplex of K spanned by the vertices a such that $a_{i-1} < a < a_i$.

Lemma 1. *Let K be an ordered triangulation of an integral* (*resp.* mod 2) *Euler space, and let $\sigma < K$. Then each σ_i is an integral* (*resp.* mod 2) *Euler space and $\chi(\sigma_i) = 1 + (-1)^{\dim \sigma_i}$* (*resp. $\chi(\sigma_i)$ is even*).

Proof. It is easy to see that $Lk(\sigma, K)$ is an Euler space, and hence that each σ_i is an Euler space. Since σ_i is the link of a maximal simplex in the join of the remaining σ_j's, it follows that σ_i has the Euler characteristic of a sphere of the same dimension (cf. [2, §2]).

Lemma 2. *If K is an ordered triangulation of a compact integral (resp. mod 2) Euler space, then $\chi(K) = \Sigma_{a \in K}(-1)^{|a|}$ (resp. $\chi(K) \equiv (\#(K) \pmod 2)$).*

Proof. Note that

$$\begin{aligned}\chi(K) &= \sum_p (-1)^p(\text{number of } p\text{-simplices } \langle a_0 \cdots a_p\rangle)\\ &= \sum_a \left\{\sum_{p \geq 1} [(-1)^p(\text{number of } (p-1)\text{-simplices in } a^-)] + 1\right\}\\ &= \sum_a (1 - \chi(a^-)).\end{aligned}$$

But by Lemma 1, $1 - \chi(a^-) = (-1)^{|a|}$ (resp. $1 - \chi(a^-) \equiv 1 \pmod 2$).

Lemma 3. *If K is an ordered triangulation of an integral Euler space, then*

$$\partial s_p(K) = (1 + (-1)^{n-p})s_{p-1}(K).$$

Proof. By Lemmas 1 and 2, this is the same as the proof of Proposition 1 of [2], replacing $Lk(a_i, a_{i+1})$ by σ_{i+1}.

Proposition. *Let K be an ordered triangulation of X.*

(i) *If X is a* mod-2 *Euler space, then $s_p(K)$ represents $w_p(X) \in H_p(X, \mathbf{Z}_2)$.*

(ii) *Moreover, if X is an integral Euler space and $n - p$ is odd, $p > 0$, then $s_p(K)$ represents $w_p(X) \in H_p(X, \mathbf{Z})$.*

Proof. By Lemma 3, (ii) follows from (i). To prove (i), define a simplicial map ϕ: $K' \to K$ as follows. If $\sigma < K$ and $\hat{\sigma}$ denotes its barycenter, let $\phi(\hat{\sigma}) = m(\sigma)$ where $m(\sigma)$ is the maximum vertex of σ. ϕ extends to a simplicial map that induces the identity in homology.

Now an easy computation gives ($\mathbf{Z}_2$ coefficients!)

$$\phi(s_p(K')) = \sum_{a_0 < \cdots < a_p} \alpha_{a_0 \cdots a_p} \langle a_0 \cdots a_p\rangle$$

where, if $\sigma = \langle a_0 \cdots a_p\rangle$,

$$\alpha_{a_0 \cdots a_p} = (\#(\sigma_0) + 1)(\#(\sigma_1) + 1) \cdots (\#(\sigma_{p+1}) + 1).$$

But by Lemma 2, $\#(\sigma_i) \equiv \chi(\sigma_i)$, which is even by Lemma 1. Thus $a_{a_0 \cdots a_p} \equiv 1$ and $\phi(s_p(K')) = s_p(K)$.

The first part of the Proposition was the only step missing in the proof of the Theorem. To prove the Corollary, let $\times$: $C_*(K', \mathbf{Z}) \otimes C_*(L', \mathbf{Z}) \to C_*((K \times L)', \mathbf{Z})$ be any chain map inducing the cross-product in homology; i.e., such that $\partial(x \times y) = (\partial x) \times y + (-1)^p x \times \partial y$ if $x \in C_p(K', \mathbf{Z})$. For example, we could use the same formula (2) with appropriate signs.

An easy computation then shows that if $m + n - p$ is odd,

$$\partial\left(\sum_{r=0}^{p+1} s_r(K') \times s_{p+1-r}(L')\right) = 2 \sum_{r=0}^{p} (-1)^{mr} s_r(K') \times s_{p-r}(L').$$

Thus $\Sigma_{r=0}^{p}(-1)^{mr} s_r(K') \times s_{p-r}(L')$ is an integral cycle that represents the Bockstein of $w_{p+1}(X \times Y)$.

REFERENCES

1. E. Akin, *Stiefel-Whitney homology classes and bordism*, Trans. Amer. Math. Soc. (to appear).

2. S. Halperin and D. Toledo, *Stiefel-Whitney homology classes*, Ann. of Math. (2) 96 (1972), 511–525. MR 47 #1072.

3. D. Sullivan, *Combinatorial invariants of analytic spaces*, Proc. Liverpool Singularities Sympos. I (1969/70), Lecture Notes in Math., vol. 192, Springer-Verlag, Berlin, 1971, pp. 165–168. MR 43 #4063.

4. H. Whitney, *On the theory of sphere bundles*, Proc. Nat. Acad. Sci. U.S.A. 26 (1940), 148–153. MR 1, 220.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO, CANADA (Current address of Stephen Halperin)

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540

Current address (Domingo Toledo): Department of Mathematics, Columbia University, New York, New York 10027