AUTOMORPHISMS COMPATIBLE WITH GROUP ACTIONS

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ABSTRACT. Let A be a group of bijections of the set $\mathcal{O}$ and let G be a subgroup of A . Our purpose here is to introduce several other subgroups of A consisting of bijections compatible with the G -orbit structure on $\mathcal{O}$ and to investigate these groups when $\mathcal{O}$ is the Grassmann variety of all p -dimensional subspaces of a vector space V and G is a group of automorphisms of the Grassmannian induced by a group of linear transformations on V .

1. Introduction and preliminaries. Let A be a group of bijections of the set $\mathcal{O}$ and let G be a subgroup of A . Our purpose here is to introduce several other subgroups of A consisting of bijections compatible with the G -orbit structure on $\mathcal{O}$, and to investigate these groups when $\mathcal{O}$ is the Grassmann variety of all p -dimensional subspaces of a vector space V and G is a group of automorphisms of the Grassmannian induced by a group of linear transformations on V . The problem just mentioned is connected with the determination of duality theories for metabelian Lie algebras [3, §3].

If G and A are as above, the group $\mathcal{F}$ consisting of elements of A fixing each G -orbit is called the orbit fixing group of G in A . The subgroup $\mathcal{S}$ of A consisting of all elements of A taking G -orbits to G -orbits is called the orbit shuffling group of G in A . Then $G \subseteq \mathcal{F} \subseteq \mathcal{S} \subseteq A$, and $\mathcal{S}$ consists of those elements of A inducing bijections on the orbit space $\mathcal{O}/G$. Since $\mathcal{F}$ is the kernel of the natural map $\mathcal{S} \rightarrow \text{Bij}(\mathcal{O}/G)$, we see that $\mathcal{F}$ is normal in $\mathcal{S}$. The main result achieved here is the

Theorem. *Suppose G is a connected, irreducible algebraic subgroup of $\text{GL}(V)$ containing the nonzero scalars. Let $\mathcal{S}, \mathcal{F}$ be the orbit shuffling and fixing groups for the induced action of G on the projective Grassmann variety of all p -dimensional subspaces of V . Then $\mathcal{S}/\mathcal{F}$ is a finite group.*

This result indicates that the behavior observed in [3] is part of a more general phenomena.

Suppose that $\mathcal{O}$ is a set with an equivalence relation, and there is a group G acting on $\mathcal{O}$ whose orbits coincide with equivalence classes. We

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avoid mentioning what the group A should be. What choice one makes usually depends on what structure $\mathcal{U}$ might have in addition to being a set. For example, if $\mathcal{U}$ were a variety, G an algebraic group and $\mathcal{U} \times G \rightarrow \mathcal{U}$ a morphism, one might choose A to be the "algebraic" automorphisms of $\mathcal{U}$. A typical problem is to find a "canonical form" for elements of $\mathcal{U}$ —a "canonical" member of each equivalence class (orbit). Were $\mathcal{F}$ bigger than G it would afford one more transformation to manipulate elements of $\mathcal{U}$ and it could possibly simplify the problem.

$\mathcal{S}$ by itself is apparently not as significant. It is a simple observation, however, that $\mathcal{S}/\mathcal{F}$ is isomorphic to the group of bijections on $\mathcal{U}/G$ induced by $\mathcal{S}$. In connection with the duality theory problem mentioned above, the group $\mathcal{S}/\mathcal{F}$ is in bijective correspondence with the collection of all duality theories satisfying certain axioms (see [3, §1]).

Proposition 1. *Let A be a group of bijections of the set $\mathcal{U}$ and let G be a subgroup of A . Let $\mathcal{F}$ and $\mathcal{S}$ be the orbit fixing and shuffling groups respectively. Then $\mathcal{S}$ is the normalizer of $\mathcal{F}$ in A .*

Proof. We have seen already that $\mathcal{F}$ is normal in $\mathcal{S}$. Suppose τ in A normalizes $\mathcal{F}$. Let v_1, v_2 be in $\mathcal{U}$ and suppose θ in G satisfies $v_1\theta = v_2$. Then $v_2\tau = v_1\theta\tau = v_1\tau\theta'$ for some θ' in $\mathcal{F}$. But θ' fixes G -orbits so there is a θ'' in G with $v_1\theta'' = v_1\tau\theta' = v_2\tau$. Thus τ shuffles G -orbits.

2. **Automorphisms compatible with group actions on Grassmann varieties.** Let K be an algebraically closed field of characteristic zero and let V be a finite-dimensional K -vector space. Suppose G is a connected algebraic subgroup of $GL(V)$ containing the nonzero scalars. For any $p \leq \dim V$ there is an induced action of G on G_p —the projective Grassmann variety of p -dimensional subspaces of V . G_p is the projectivization of the homogeneous affine subvariety D_p of $\Lambda^p V$ consisting of all decomposable p -vectors. Consider the diagram

$$\begin{array}{ccccccc}
 & & & & GL(\Lambda^p V) & & \\
 & & & & | & & \\
 A(G_p) & \xleftarrow{\pi} & St(D_p) & \xleftarrow{\Lambda^p} & GL(V) & & \\
 | & & | & & | & & \\
 \mathcal{S} & \xleftarrow{\quad} & \mathcal{S}_1 & \xleftarrow{\quad} & \mathcal{S}_2 & & \\
 | & & | & & | & & \\
 \mathcal{F} & \xleftarrow{\quad} & \mathcal{F}_1 & \xleftarrow{\quad} & \mathcal{F}_2 & & \\
 | & & | & & | & & \\
 \mathcal{G} = \pi\Lambda^p G & \xleftarrow{\quad} & \Lambda^p G & \xleftarrow{\quad} & G & &
 \end{array}$$

where $A(G_p)$ is the group of "algebraic" automorphisms of G_p , $\text{St}(D_p)$ is the stabilizer of D_p in $\text{GL}(\Lambda^p V)$, π is the natural map, and Λ^p is the representation of $\text{GL}(V)$ on $\Lambda^p V$ given on decomposables by

$$\Lambda^p(\theta)(v_1 \wedge \dots \wedge v_p) = \theta(v_1) \wedge \dots \wedge \theta(v_p).$$

In addition, $\mathcal{S}$ and $\mathcal{F}$ are the orbit shuffling and fixing groups of $\mathcal{G}$ in $A(G_p)$, $\mathcal{S}_1 = \pi^{-1}(\mathcal{S})$, $\mathcal{F}_1 = \pi^{-1}(\mathcal{F})$, $\mathcal{S}_2 = (\Lambda^p)^{-1}(\mathcal{S}_1)$, and $\mathcal{F}_2 = (\Lambda^p)^{-1}(\mathcal{F}_1)$. See [2, §§1, 2] and [3, §1] for further details.

In [2, §2] it is shown that π is surjective and the kernel of π consists of nonzero scalars. According to Westwick's theorem [4, p. 1127], $\text{St}(D_p) = \Lambda^p(\text{GL}(V))$ except when $\dim V/2 = p$, and in this case $(\text{St}(D_p): \Lambda^p(\text{GL}(V))) = 2$.

For the purpose of using Lie algebras later on, we indicate here that all the groups in the diagram are algebraic groups. Since K is algebraically closed, $\Lambda^p G$ and $\mathcal{G}$ are algebraic as images of the group G by rational homomorphisms. Let τ denote the natural map from D_p to G_p . The formula $\Lambda^p(\alpha|_V) = \alpha|_{\Lambda^p V}$, and the fact that K is algebraically closed shows that $\Lambda^p(G)$ also contains the nonzero scalars. Thus if v is a nonzero decomposable p -vector the projectivization of the orbit of v by $\Lambda^p(G)$ is the orbit of $\tau(v)$, and conversely, the inverse image under τ of the orbit of $\tau(v)$ is the orbit of v by $\Lambda^p(G)$. Thus $\mathcal{F}_1$ and $\mathcal{S}_1$ are the orbit fixing and shuffling groups of $\Lambda^p(G)$ in $\text{St}(D_p)$.

Whenever G is a connected algebraic group acting on a variety $\mathcal{U}$ by a morphism $\mathcal{U} \times G \rightarrow \mathcal{U}$, orbits are locally closed subvarieties [1, p. 98]. Moreover, the closure of a nonclosed orbit is the disjoint union of the orbit itself together with orbits of strictly smaller dimensions. Thus any algebraic automorphism of $\mathcal{U}$ fixes (shuffles) orbits if and only if it fixes (shuffles) closures of orbits.

According to [1, p. 97] $\text{St}(D_p)$ is algebraic as it is the stabilizer of the closed set D_p . And since $\Lambda^p G$ is connected algebraic, $\mathcal{F}_1$ and $\mathcal{S}_1$ are algebraic as intersections of transporters of closed sets (i.e. closures of orbits). Also $A(G_p)$, $\mathcal{F}$, and $\mathcal{S}$ are algebraic since they are images of $\text{St}(D_p)$, $\mathcal{F}_1$, and $\mathcal{S}_1$ by the rational homomorphism π . According to Westwick's theorem $(\mathcal{S}_1: \Lambda^p \mathcal{S}_2) \leq 2$, $(\mathcal{F}_1: \Lambda^p \mathcal{F}_2) \leq 2$. Thus $\Lambda^p \mathcal{S}_2$ and $\Lambda^p \mathcal{F}_2$ are closed (algebraic) and their inverse images $\mathcal{S}_2$ and $\mathcal{F}_2$ are closed in $\text{GL}(V)$, hence algebraic.

Notation. For any algebraic group H we let $L(H)$ be its Lie algebra.

Proposition 2. (1) $L(\mathcal{F}_2)$ is an ideal in $L(\mathcal{S}_2)$.

(2) $(\mathcal{S}: \mathcal{F}) \leq 2$ $(\mathcal{S}_2: \mathcal{F}_2)$.

Proof. (1) $\mathcal{S}$ normalizes $\mathcal{F}$ and π is always surjective so $\mathcal{S}_1$ normalizes $\mathcal{F}_1$. Thus $L(\mathcal{S}_1)$ normalizes $L(\mathcal{F}_1)$. At worst $(\text{St}(D_p): \Lambda^p(\text{GL}(V))) = 2$, hence $(\mathcal{S}_1: \Lambda^p \mathcal{S}_2) \leq 2$, $(\mathcal{F}_1: \Lambda^p \mathcal{F}_2) \leq 2$. Therefore $d\Lambda^p$ (the differential) maps $L(\mathcal{S}_2)$ onto $L(\mathcal{S}_1)$ and $L(\mathcal{F}_2)$ onto $L(\mathcal{F}_1)$. As a result $L(\mathcal{S}_2)$ normalizes $L(\mathcal{F}_2)$.

(2) Since π is surjective $(\mathcal{S}: \mathcal{F}) \leq (\mathcal{S}_1: \mathcal{F}_1)$. Also, since $(\mathcal{S}_1: \Lambda^p \mathcal{S}_2) \leq 2$, $(\mathcal{F}_1: \Lambda^p \mathcal{F}_2) \leq 2$, one easily sees that

$$(\mathcal{S}_1: \mathcal{F}_1) \leq 2(\Lambda^p \mathcal{S}_2: \Lambda^p \mathcal{F}_2) \leq 2(\mathcal{S}_2: \mathcal{F}_2)$$

Theorem 3. Suppose G is a connected, irreducible algebraic subgroup of $\text{GL}(V)$ containing the nonzero scalars. Let $\mathcal{S}, \mathcal{F}$ be the orbit shuffling and fixing groups for the induced action of G on the projective Grassmann variety of all p -dimensional subspaces of V . Then $\mathcal{S}/\mathcal{F}$ is a finite group.

Proof. By (2) of the proposition it suffices to show that $\mathcal{S}_2/\mathcal{F}_2$ is a finite group. Since $\mathcal{S}_2$ and $\mathcal{F}_2$ are algebraic groups, it suffices to show $L(\mathcal{S}_2) = L(\mathcal{F}_2)$.

Since G acts irreducibly on V so does $L(G)$. In fact $L(\mathcal{F}_2)$ acts irreducibly since it contains $L(G)$. Let $L = L(\mathcal{F}_2)$. Then $L = [L, L] \oplus Z(L)$ ($Z(L)$ = center of L), $[L, L]$ is semisimple, and $Z(L) = KI_V$ since L acts irreducibly. By part (1) of the proposition, $L(\mathcal{S}_2)$ normalizes $L = [L, L] \oplus KI_V$, hence it normalizes $[L, L]$. Since $[L, L]$ is a semisimple ideal in $L(\mathcal{S}_2)$, $L(\mathcal{S}_2)$ splits as the direct sum of $[L, L]$ with its centralizer in $L(\mathcal{S}_2)$. Since L acts irreducibly and $L = [L, L] \oplus KI_V$, $[L, L]$ also acts irreducibly. Thus its centralizer in $L(\mathcal{S}_2)$ must consist only of scalars. Hence $L(\mathcal{S}_2) = [L, L] \oplus KI_V = L$.

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