A UNIFORM SET COVERING LEMMA¹

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ABSTRACT. The bounded set system $H = (V, \mathcal{F})$ is composed of a nonvoid set V and a set, $\mathcal{F}$, of nonvoid subsets of V , a finite number of which cover V . $C \subset V$ is a critical subset of H if every proper subset of C requires fewer members of $\mathcal{F}$ to cover it than are needed to cover C . For $|\mathcal{F}|$ finite, it is shown that every $A \subset V$ contains a critical $C \subset A$ requiring the same number of members of $\mathcal{F}$ in a minimum cover. For $v \in V$, $l(v)$ is the largest number of members of $\mathcal{F}$ in any minimum cover of any critical set containing v . For $|\mathcal{F}|$ finite, it is shown that there exists a covering $A_1, A_2, \dots, A_k, A_i \in \mathcal{F}$ for $1 \leq i \leq k$, such that $v \in \bigcup_{i=1}^{l(v)} A_i$ for all $v \in V$. An application to graph coloring is described.

1. Coverings and critical sets. The set system (or hypergraph [1]) $H = (V, \mathcal{F})$ is composed of a nonvoid set V and a set, $\mathcal{F}$, of nonvoid subsets of V that cover V . $H = (V, \mathcal{F})$ is a bounded set system if a finite number of members of $\mathcal{F}$ cover V , and H is a finite set system if $|V|$ is finite. A set system is necessarily bounded if either $|V|$ or $|\mathcal{F}|$ is finite. For the bounded set system $H = (V, \mathcal{F})$, the covering number, $\chi_H(A)$, of $A \subset V$ is the minimum number of members of $\mathcal{F}$ whose union contains A , and $A \subset V$ is k -partite in H if $\chi_H(A) \leq k$. The subset $C \subset V$ is critical in the bounded set system H if $\chi_H(A) \leq \chi_H(C) - 1$ for all $A \subset C, A \neq C$. Note that $\{v\}$ is critical for every $v \in V$.

For a finite set system $H = (V, \mathcal{F})$ it is clear that if $\chi_H(A) = n$, then there exists a critical $C \subset A$ such that $\chi_H(C) = \chi_H(A) = n$. This result is still obtained if only $\mathcal{F}$ is assumed finite.

Lemma 1. For the set system $H = (V, \mathcal{F})$ where $|\mathcal{F}| = q$, any $A \subset V$ contains a critical $C \subset A$ with $\chi_H(C) = \chi_H(A)$ where

$$|C| \leq \binom{q}{\chi_H(A) - 1}.$$

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Proof. Let $H = (V, \mathcal{F})$ with $|\mathcal{F}| = q$. Then for $1 \leq k \leq q$, there are at most $\binom{q}{k}$ maximal k -partite sets in H , so every k -partite set of H is contained in a maximal k -partite set of H . For $A \subset V$ with $\chi(A) = 1$, $\{v\}$ is critical for any $v \in A$, thus satisfying the lemma in this case. Let $A_0 \subset V$ be chosen with $\chi_H(A_0) = k \geq 2$. Choose B_1 to be one of the maximal $(k-1)$ -partite sets of H such that $A_0 \cap B_1$ is maximal in A_0 with $\chi(A_0 \cap B_1) = k-1$. Choose $a_1 \in A_0 - B_1$, and let $A_1 = A_0 \cap B_1 \cup \{a_1\}$, so then $\chi_H(A_1) = k$. If $\chi_H(\{a_1, a_2, \dots, a_i\}) \leq k-1$, for $i \geq 1$, choose B_{i+1} to be one of the maximal $(k-1)$ -partite sets of H with $\{a_1, a_2, \dots, a_i\} \subset B_{i+1}$, where also $A_i \cap B_{i+1}$ is maximal in A_i with $\chi_H(A_i \cap B_{i+1}) = k-1$. Choose $a_{i+1} \in A_i - B_{i+1}$, and let $A_{i+1} = A_i \cap B_{i+1} \cup \{a_{i+1}\}$, so then $\chi_H(A_{i+1}) = k$. Thus when $\chi_H(\{a_1, a_2, \dots, a_i\}) \leq k-1$, $A_0 \supset A_1 \supset A_2 \supset \dots \supset A_i \supset A_{i+1} \supset \{a_1, a_2, \dots, a_i, a_{i+1}\}$ and $\chi_H(A_j) = k$ for $0 \leq j \leq i+1$. Furthermore, the B_j so chosen satisfy $\{a_1, a_2, \dots, a_{j-1}\} \subset B_j$, $a_j \notin B_j$, for $1 \leq j \leq i+1$, so the B_j are distinct maximal $(k-1)$ -partite sets of H . Eventually for some index $n \leq \binom{q}{k-1} - 1$, $\chi_H(\{a_1, a_2, \dots, a_{n+1}\}) = k$. Let $C = \{a_1, a_2, \dots, a_{n+1}\}$. Then $\chi_H(C) = k$ and $\chi_H(C - \{a_{n+1}\}) = k-1$. For $1 \leq j \leq n$, $\{a_1, a_2, \dots, a_{j-1}\} \subset B_j$. Also $\{a_{j+1}, a_{j+2}, \dots, a_{n+1}\} \subset A_{n+1} \subset A_j \subset A_{j-1}$, and since $A_j = A_{j-1} \cap B_j \cup \{a_j\}$, it follows that $\{a_{j+1}, a_{j+2}, \dots, a_{n+1}\} \subset B_j$. Hence $C - \{a_j\} \subset B_j$, and $\chi_H(C - \{a_j\}) \leq \chi_H(B_j) = k-1$ for $1 \leq j \leq n$. So $C = \{a_1, \dots, a_{n+1}\}$ is critical with $\chi_H(C) = \chi_H(A)$, and $|C| = n+1 \leq \binom{q}{k-1}$. $\square$

Corollary 1.1. Let $H = (V, \mathcal{F})$ be a set system with $|\mathcal{F}| = q$. Then every critical $C \subset V$ has at most

$$\binom{q}{\chi_H(C) - 1}$$

members. Furthermore, for any $1 \leq k \leq q$, there exists an $H = (V, \mathcal{F})$ with $|\mathcal{F}| = q$ and a critical $C \subset V$ with $\chi_H(C) = k$ and

$$|C| = \binom{q}{\chi_H(C) - 1}.$$

Proof. Let $H = (V, \mathcal{F})$ with $|\mathcal{F}| = q$. For any critical $C' \subset V$, by Lemma 1 there exists $C \subset C'$, C critical, $\chi_H(C) = \chi_H(C')$ and

$$|C| \leq \binom{q}{\chi_H(C) - 1}.$$

Since C' is critical with $\chi_H(C) = \chi_H(C')$, $C = C'$, verifying the bound of the lemma. To show that this result is best possible, let k and q be given with $1 \leq k \leq q$. Let $H = (V, \mathcal{F})$ be constructed so that $|V| = \binom{q}{k-1}$, where $\mathcal{F} = \{S_1, S_2, \dots, S_q\}$, and each member of V is contained in a unique choice of $q - k + 1$ of the subsets $\{S_i\}$, $i = 1, \dots, q$. Thus any union of $k - 1$ of the q members of $\mathcal{F}$ will contain all but one member of V , and $V - \{v\}$ will be covered by a unique set of $k - 1$ of the q members of $\mathcal{F}$ for any $v \in V$. Hence $\chi_H(V) = k$ and $\chi_H(V - \{v\}) = k - 1$ for any $v \in V$, so V is critical in H with

$$|V| = \binom{q}{\chi_H(V) - 1},$$

showing the bound is best possible. $\square$

It should be noted that the result of Lemma 1 is not obtained for all bounded set systems. For example, let $H' = (V', \mathcal{F}')$, where

$$(1) \quad V' = \{0, 1, 2, \dots\}, \quad \mathcal{F}' = \{E, D, I_1, I_2, I_3, \dots\},$$

where $E = \{0, 2, 4, 6, \dots\}$, $D = \{1, 3, 5, \dots\}$, and $I_j = \{0, 1, 2, \dots, j\}$ for $j \geq 1$. Any nonempty set of nonnegative integers which are all even, all odd, or finite in number will have a covering number of unity in H' , and all other nonempty sets of nonnegative integers will have a covering number of two in H' . Any $\{i\}$, $i \geq 0$, is critical in H' with covering number unity, but there are no critical sets with covering number two in H' .

II. The covering level and uniform set coverings. For any bounded set system $H = (V, \mathcal{F})$, each element $v \in V$ has a *covering level*, $l(v)$, given by

$$(2) \quad l(v) \equiv \max_{C \subset V} \{\chi_H(C) \mid v \in C, C \text{ is critical}\},$$

where then $1 \leq l(v) \leq \chi_H(V)$ for all $v \in V$.

The following lemma shows that in any bounded set system where $|\mathcal{F}|$ is finite, the covering level of v is equivalent to the largest covering number of those sets in which v is an "essential member."

Lemma 2. *Let the set system $H = (V, \mathcal{F})$ where $|\mathcal{F}|$ is finite. For any $v \in V$,*

$$(3) \quad l(v) = \max_{A \subset V} \{\chi_H(A) \mid v \in A, \chi_H(A - \{v\}) \leq \chi_H(A) - 1\}.$$

Proof. Let the set system $H = (V, \mathcal{F})$ where $|\mathcal{F}|$ is finite, so H is bounded. For every critical $C \subset V$ with $v \in C$, we have $\chi_H(C - \{v\}) \leq \chi_H(C) - 1$, so $l(v) \leq \max_{A \subset V} \{\chi_H(A) \mid v \in A, \chi_H(A - \{v\}) \leq \chi_H(A) - 1\}$. Let $A^* \in V$ be a set

maximizing the right-hand side of (3). If A^* is not critical, from Lemma 1 there exists a critical $C \subset A^*$ with $\chi_H(C) = \chi_H(A^*)$. If $v \notin C$, then $C \subset A^* - \{v\}$ would imply $\chi_H(C) \leq \chi_H(A^* - \{v\}) \leq \chi_H(A^*) - 1$, a contradiction. Thus v is contained in the critical set C , so $l(v) \geq \chi_H(C) = \chi_H(A^*) = \max_{A \subset V} \{\chi_H(A) | v \in A, \chi_H(A - \{v\}) \leq \chi_H(A) - 1\}$, proving the lemma. $\square$

Corollary 2.1. *Let the set system $H = (V, \mathcal{F})$ where $|\mathcal{F}|$ is finite. Let $v \in V$ have $l(v) = k$, and let $B_k \subset V$ be any maximal k -partite set of H . Then $v \in B_k$.*

Proof. For v and B_k as specified, assume $v \notin B_k$. Then $\chi_H(B_k \cup \{v\}) = k + 1$, since B_k is maximal with $\chi_H(B_k) \leq k$. Thus, from Lemma 2, $l(v) \geq k + 1$, a contradiction. So $v \in B_k$. $\square$

From Corollary 2.1 it may be concluded that when $|\mathcal{F}|$ is finite, $l(v) = 1$ if and only if v is in every maximal member of $\mathcal{F}$.

For the bounded set system $H = (V, \mathcal{F})$, there must exist some covering $A_1, A_2, \dots, A_k$ of V where $A_i \in \mathcal{F}$ for $1 \leq i \leq k$, and $k = \chi_H(V)$. When $|\mathcal{F}|$ is finite, the A_i may each be assumed to be maximal members of $\mathcal{F}$, but this does not guarantee that $B = A_1 \cup A_2$ is a maximal 2-partite set of H . The following lemma shows that some $\chi_H(V)$ covering of V yields initial segment unions which are maximal k -partite sets for all $1 \leq k \leq \chi_H(V)$ whenever $|\mathcal{F}|$ is finite.

Lemma 3 (Uniform set covering lemma). *Let the set system $H = (V, \mathcal{F})$ where $|\mathcal{F}|$ is finite. Then there exists a minimum covering $A_1, A_2, \dots, A_{\chi_H(V)}$ of V , $A_i \in \mathcal{F}$ for $1 \leq i \leq \chi_H(V)$, such that*

- (i) $\bigcup_{i=1}^k A_i$ is a maximal k -partite set for $1 \leq k \leq \chi_H(V)$,
- (ii) $v \in \bigcup_{i=1}^{l(v)} A_i$ for all $v \in V$.

Proof. For $\chi_H(V) = 1$, the result of the lemma is immediate. Assume condition (i) of the lemma to hold for all set systems $H = (V, \mathcal{F})$ with finite $|\mathcal{F}|$ and covering numbers $\chi_H(V) \leq n - 1$ for a given $n \geq 2$, and let $H^* = (V^*, \mathcal{F}^*)$ have finite $|\mathcal{F}^*|$ and covering number $\chi_{H^*}(V^*) = n$. Let $A_1^*, A_2^*, \dots, A_n^*, A_i^* \in \mathcal{F}^*$, for $1 \leq i \leq n$, be a minimum covering of V^* . Let B be a maximal $(n - 1)$ -partite set of H^* such that $\bigcup_{i=1}^{n-1} A_i^* \subset B \subset V^*$. Form the set system $H_B = (B, \mathcal{F}_B)$, where $\mathcal{F}_B = \{S | S \neq \emptyset, S = S' \cap B \text{ for some } S' \in \mathcal{F}^*\}$. Then $\chi_{H_B}(B) = n - 1$, and $|\mathcal{F}_B|$ is finite, so by our initial assumption there exists a covering $B_1, B_2, \dots, B_{n-1}$ of B with $B_i \in \mathcal{F}_B$ and $B_i = S_i \cap B$ for $S_i \in \mathcal{F}^*$

for $1 \leq i \leq n-1$ where $\bigcup_{i=1}^k B_i$ is a maximal k -partite set of B for $1 \leq k \leq n-1$. Then $B = \bigcup_{i=1}^{n-1} B_i \subset \bigcup_{i=1}^{n-1} S_i$, and $\bigcup_{i=1}^{n-1} S_i$ is an $(n-1)$ -partite set of H^* . However, B was chosen to be a maximal $(n-1)$ -partite set of H^* , so $B = \bigcup_{i=1}^{n-1} S_i$, and $B_i = S_i$ for all i . For $1 \leq k \leq n-1$, let $\bigcup_{i=1}^k B_i \subset D_k \subset V^*$ where D_k is a maximal k -partite set of H^* . Now $D_k \cap B$ is a k -partite set of H_B , and $D_k \cap B = \bigcup_{i=1}^k B_i$, since $\bigcup_{i=1}^k B_i$ is a maximal k -partite set of H_B . Also $D_k \cap V - B = \emptyset$, since otherwise $D_k \cup \bigcup_{i=k+1}^{n-1} B_i$ would be an $(n-1)$ -partite set of H^* strictly containing B . Thus $\bigcup_{i=1}^k B_i$ is a maximal k -partite set of $H^* = (V^*, \mathcal{F}^*)$ for $1 \leq k \leq n-1$, hence $B_1, B_2, \dots, B_{n-1}, A_n$ is a cover of V^* satisfying condition (i) of the statement of the lemma. By induction, condition (i) then holds for all set systems $(V, \mathcal{F})$ with $|\mathcal{F}|$ finite. Condition (ii) then follows from condition (i) and Corollary 2.1, proving the lemma.

Let $A_1, A_2, \dots, A_k$ be a covering of V for the bounded set system $H = (V, \mathcal{F})$. Any set $A \subset V$ must meet at least $\chi_H(A)$ members of this covering, and in particular if C_v is a critical subset of V containing v with $l(v) = \chi(C_v)$, then at least $l(v)$ members from $A_1, A_2, \dots, A_k$ are needed to cover C_v . This motivates our definition that a covering $A_1, A_2, \dots, A_k$ of V for the set system $H = (V, \mathcal{F})$ is a *uniform set covering* for H if $v \in \bigcup_{i=1}^{l(v)} A_i$ for all $v \in V$. Lemma 3 thus assures us that a uniform set covering always exists for any set system $H = (V, \mathcal{F})$ with finite $|\mathcal{F}|$. The bounded set system $H' = (V', \mathcal{F}')$ described in equation (1) has $l(v) = 1$ for all $v \in V'$, yet $\chi_{H'}(V') = 2$. Thus H' has no uniform set covering, so all bounded set systems do not possess uniform set coverings.

A uniform set covering provides a minimum set covering for the whole of V . Conversely, it is evident for any finite set system H that the methods employed in the inductive proof of Lemma 3 could be utilized recursively along with any existing procedure for determining a minimum set covering of a finite set to yield a procedure for determining a uniform set covering of H . Thus the problem of determining a uniform set covering of a finite set system clearly has computational complexity which is equivalent, in the sense of existence or nonexistence of a polynomial bounded algorithm, to that of determining a minimum set covering of a finite set, and therefore belongs to a large class of finite but computationally difficult problems described in [3].

III. An application to graph theory. The uniform set covering lemma can be applied to the theory of graph colorings [5]. Given the graph $G =$

(V, E) , form the finite set system $H = (V, \mathcal{F}_I)$, where V is the finite vertex set from G , and $\mathcal{F}_I$ is the set of independent sets of vertices of the graph G . For $A \subset V$, the covering number $\chi_H(A)$ thus corresponds to the chromatic number of the induced subgraph $\langle A \rangle$ of G . The critical set C of $H = (V, \mathcal{F}_I)$ thus corresponds to a color-critical subgraph $\langle C \rangle$ of G , and $l(v)$ becomes the largest chromatic number of any color-critical subgraph of G containing v . A *Grundy function* on a graph G is an assignment of nonnegative integer values (colors) to the vertices of G such that each vertex has an assigned value equal to the lowest nonnegative integer not occurring on any adjacent vertex of G . Let $\chi(G)$ denote the chromatic number of the graph G .

Corollary 3.1. *There exists a Grundy function g on the graph $G = (V, E)$ such that for every $v \in V$,*

$$(4) \quad g(v) \leq -1 + \max_{A \subset V} \{\chi(\langle A \rangle) \mid v \in A, \langle A \rangle \text{ is color-critical}\}.$$

Proof. Applying Lemma 3 to the set system $(V, \mathcal{F}_I)$ of independent sets of the graph G , the independent sets $A_1, A_2, \dots, A_{\chi(G)}$ may be determined so that $v \in \bigcup_{i=1}^{l(v)} A_i$ for all $v \in V$. Assigning members of A_i the color value $i - 1$ for $1 \leq i \leq \chi(G)$, a graph coloring is determined where each vertex receives a color value $\leq -1 + l(v) = -1 + \max_{A \subset V} \{\chi(\langle A \rangle) \mid v \in A, \langle A \rangle \text{ is color-critical}\}$. If this color function on the vertices is not a Grundy function, color values on vertices may be lowered eventually yielding a Grundy function g of G which satisfies (4). $\square$

In [4] we derived the following result: For any graph $G = (V, E)$, there exists a Grundy function g on V such that for every vertex $v \in V$,

$$(5) \quad g(v) \leq \max\{\lambda(G') \mid G' \text{ a subgraph of } G \text{ containing } v\},$$

where $\lambda(G')$ is the edge-connectivity of G' . From a result of Dirac [2], Ore [5, p. 165] has shown that every color-critical graph G satisfies $\chi(G) \leq \lambda(G) + 1$; thus our result from [4] is contained in Corollary 3.1.

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