CONTINUATION OF RIEMANN SURFACES

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ABSTRACT. It is known that a nonplanar Riemann surface cannot be continued into all compact Riemann surfaces of a fixed positive genus. The Poincaré metric is used to construct a conformal invariant which is used to give an essentially geometric proof of this result.

1. Introduction. If there is a univalent analytic map of the Riemann surface R into the Riemann surface S , we say that R can be continued into S . A necessary condition to insure that a Riemann surface can be continued into all compact Riemann surfaces of a given positive genus was given by Heins [2].

Theorem 1. *If the Riemann surface R can be continued into all compact Riemann surfaces of given positive genus, then R is conformally equivalent to a bounded plane region.*

The proof consists of showing first that an open planar subregion of a compact Riemann surface of positive genus is conformally equivalent to a bounded plane region; second, that if for some g , $g > 1$, R can be continued into all compact surfaces of genus g then R can be continued into all compact surfaces of genus 1; and finally that any surface which can be continued into all compact surfaces of genus 1 must be planar. The proof yields a bit more than Theorem 1. It shows that if R can be continued into certain surfaces of genus g which cover surfaces of genus 1 of arbitrarily large module then R is planar.

We will exhibit a specific geometric limitation on the class of Riemann surfaces into which a given nonplanar Riemann surface can be extended. This will allow a strengthening of Theorem 1 and will yield a proof which emphasizes the essentially geometric nature of the result.

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2. **The size of a Riemann surface.** For any Riemann surface S which has the unit disk as its universal covering surface and γ a curve on S , denote by $|\gamma|_S$ the length of γ with respect to the Poincaré metric on S . Let S be such a surface. For A an annulus in S , a subset of S homeomorphic to a closed annulus, define $|A|_S = \inf\{|\gamma|_S; \gamma \text{ a curve in } A \text{ which connects the two components of the boundary of } A\}$. Say that a set of annuli $\{A_i\}$ reduces S if the A_i are disjoint annuli in S and $S \setminus \bigcup_i A_i$ is planar. Note that the number of annuli needed to reduce S equals the genus of S . For any such S we define the size of S by

$$\text{size}(S) = \sup \left\{ \inf_i |A_i|_S; \{A_i\} \text{ reduces } S \right\}.$$

If S is nonplanar and does not have the unit disk as its universal covering surface, then S is a compact surface of genus 1. Such S has a transitive group of conformal automorphisms [4, Volume I, Chapter I]. Hence, for such an S and p in S , $\text{size}(S \setminus \{p\})$ is independent of the choice of p . In this case we define $\text{size}(S)$ to be $\text{size}(S \setminus \{p\})$.

The function $\text{size}(S)$, now defined for all nonplanar S , can be used to describe a geometric constraint on continuation of Riemann surfaces.

Proposition 1. *A nonplanar Riemann surface cannot be continued into Riemann surfaces of arbitrarily large size.*

Proof. Let R be a nonplanar Riemann surface. Since a compact surface has no nontrivial continuations, we may assume R is noncompact. In particular, R carries a Poincaré metric. Since R is nonplanar we can choose γ_1 and γ_2 to be two smooth simple closed curves on R which cross exactly once. Let S be any nonplanar surface into which R can be continued and f a univalent analytic map of R into S . If S is a compact surface of genus 1 then f also maps R into a surface S' of the same size as S obtained by removing from S any point not in $f(R)$. Hence we may assume S carries a Poincaré metric. Let $\{A_i\}$ be any set of annuli which reduces S . The curve $f(\gamma_1) \cup f(\gamma_2)$ must contain a collection of arcs which form a curve in some A_i connecting the two boundary components of that A_i . To see this, note that otherwise the curve $\gamma_1 \cup \gamma_2$ would be mapped smoothly and univalently into the planar domain formed by adjoining to the planar domain $S \setminus (\bigcup_i A_i)$ the components of each A_i cut off by $f(\gamma_1) \cup f(\gamma_2)$. However, a planar domain cannot carry a pair of simple closed curves which crosses exactly once. Hence $|A_i|_S \leq |f(\gamma_1) \cup f(\gamma_2)|_S$. By the Schwarz-Pick lemma,

$|f(\gamma_1) \cup f(\gamma_2)|_S \leq |\gamma_1|_R + |\gamma_2|_R$. Thus $\inf_j \{|A_j|_S\} \leq |\gamma_1|_R + |\gamma_2|_R$. Since the choice of reducing set was arbitrary, $\text{size}(S) \leq |\gamma_1|_R + |\gamma_2|_R$. The proposition is proved.

Corollary. *A Riemann surface which can be continued into surfaces of arbitrarily large size is planar.*

3. **Surfaces of large size.** To apply the previous result it is necessary to know that surfaces of large size exist.

Proposition 2. *Let K be a positive number and g a positive integer. There is a compact Riemann surface S of genus g with $\text{size}(S) > K$.*

Figure 1

Proof. First we construct such a surface for $g \geq 2$ (for $g = 2$ the construction is summarized in Figure 1). Let D be the open unit disk of the complex plane endowed with the noneuclidean geometry given by the Poincaré metric. For $j = 1, \dots, 2g$ let $a_j = \exp(\pi j i / g)$. Pick α , $0 < \alpha < \pi/4g$. For $j = 1, \dots, 2g$, let r_j be the noneuclidean straight line in D joining $a_j \exp(i\alpha)$ to $a_j \exp(i(\pi/g - \alpha))$. For β with $\alpha < \beta < \pi/4g$, for $j = 1, \dots, 2g$, let s_j be the noneuclidean straight line in D joining $a_j \exp(i\beta)$ to $a_j \exp(-i\beta)$. These $4g$ noneuclidean straight lines divide the disk into a finite number of (not necessarily bounded) noneuclidean polygons. The central polygon is a fundamental domain for the surface we wish to construct. Let R be the component of $D \setminus \{(\bigcup_i r_i) \cup (\bigcup_j s_j)\}$ which contains the origin. For fixed small α and sufficiently small β , the size of the angle formed inside R by

the intersection of s_j and r_j is a monotone function of β which varies from values above $\pi/2$ to zero as β tends to α . Hence, for fixed small α there is a unique choice of β such that the $4g$ interior angles of the noneuclidean polygon $\bar{R}$ are each of size $\pi/2g$. For each α we will use this value of β . $\bar{R} = \bar{R}(\alpha)$ is a noneuclidean $4g$ -gon, the interior angles of which sum to 2π and such that the j th and $(j+2)$ nd sides are the same length for each j . Hence [4, volume II, p. 84] $\bar{R}(\alpha)$ is the fundamental region for $S(\alpha)$, a compact Riemann surface of genus g . We now show that for small α , $\text{size}(S(\alpha)) > K$. Pick and fix a small positive number ϵ . For $j = 1, 3, \dots, 2g-1$, let t_j be the noneuclidean straight line which connects $a_j \exp(i\epsilon)$ to $a_j \exp(-i\epsilon)$. For small α the part of $\bar{R}(\alpha)$ separated from the origin by the t_j corresponds to a collection of g annuli $A_1, A_3, \dots, A_{2g-1}$ on $S(\alpha)$. For $j = 1, 3, \dots, 2g-1$, let A'_j be the annulus on $S(\alpha)$ obtained by removing from A_j the intersection of A_j with a small disk about the point of intersection of s_j and r_j . The A'_j are disjoint and this set of annuli reduces $S(\alpha)$. To show $S(\alpha) > K$ it suffices to show $|A'_i|_{S(\alpha)} > K$. However for fixed ϵ , $|A_i|_{S(\alpha)}$ tends to infinity as α approaches 0.

A similar construction gives surfaces of genus 1 of arbitrarily large size.

Corollary. *Theorem 1.*

Proof. Propositions 1 and 2 show that R must be planar. The preliminary lemma of Heins' proof [2, Lemma 1] states that a Riemann surface which can be realized as an open planar subregion of a compact Riemann surface of positive genus is conformally equivalent to a bounded plane region.

4. Notes. A. Heins actually proved that surfaces which can be extended to certain highly symmetric surfaces must be planar. We have proved more than this. The surfaces of large size constructed in the proof of Proposition 2 are symmetric, however $\text{size}(S)$ is a continuous function of the conformal structure of S . More specifically, for a compact Riemann surface, the function $\text{size}(\cdot)$ is a continuous function on $T(S)$, the Teichmüller space of S . This follows from the realization of quasiconformal deformations of S as quasiconformal deformations of the associated group of linear fractional maps of the disk [1, Chapter VI]. The surfaces which have nontrivial conformal automorphisms form a subset of $T(S)$ which has a dense complement [3]. These facts, together with Proposition 2, show that there are compact surfaces of arbitrarily large size which have no nontrivial conformal automorphisms.

B. The proof of Proposition 1 does not really use the univalence of the map f . All that is used is that if γ_1 and γ_2 are curves which represent a

pair of homology classes with nonzero intersection number, then $f(\gamma_1)$ and $f(\gamma_2)$ have the same property. Hence Proposition 1, and thus also Theorem 1, apply to a weaker notion of continuation of surfaces—one defined in terms of the action of the continuation map on the homology group. For example, Theorem 1 remains true if the “continuation” map is only required to be an analytic map which has the same action on the homotopy group as a (not necessarily analytic) univalent map.

C. The idea in the proof of Theorem 1 is that there are surfaces on which a pair of homology classes with nonzero intersection number cannot consist entirely of curves which are short with respect to the Poincaré metric. Analogous results can be proved using other geometric criteria which insure that such a pair of homology classes cannot consist entirely of curves which are short in some general sense. We outline briefly a specific version of such an approach. This version is derived from a recent proof of Heins’ theorem by Professor J. Jenkins [5].

For an annulus A in a Riemann surface S , let $L(A)$ be the extremal length of the family of locally rectifiable curves in A which connect the two boundary components of A . Define $\text{size}'(S) = \sup \{ \inf_i L(A_i); \{A_i\} \text{ reduces } S \}$. The analogs of Propositions 1 and 2 are both true for this new size function. The proof of the analog of Proposition 1 is similar to the proof of Proposition 1 although some technical complications arise. The analog of Proposition 2 is proved by constructing the appropriate surfaces. Let T_g be the Riemann sphere with disjoint disks about the points $0, 1, \dots, g$ removed. Let S_g be the double of T_g . S_g is a compact Riemann surface of genus g and $\text{size}'(S_g)$ can be made arbitrarily large by making the excised disks sufficiently small. Similar constructions give surfaces S of infinite genus of various topological types with $\text{size}'(S)$ arbitrarily large size. Thus a stronger version of Proposition 2 is available for the function $\text{size}'(\cdot)$.

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