AN EXTENSION OF A THEOREM OF BESSAGA

CHARLES A. RILEY¹

ABSTRACT. In [1], C. Bessaga has shown that if X is a linear topological space admitting a weak incomplete norm w and $A \subset X$ is closed in the w -completion of X , then A is negligible in X . The present paper establishes this result in a space admitting a weak incomplete linear metric.

In his proof of the theorem mentioned above, Bessaga translates A to $A + e$, where e is in the w -completion of X , but outside X . We accomplish the same motion in the proof below. Unfortunately the elegance of Bessaga's proof is lost, since the broken line he proposes cannot be constructed in all spaces. A paranorm w in a linear space is a function to the nonnegative reals satisfying:

- (1) $w(0) = 0$,
- (2) $w(-x) = w(x)$,
- (3) $w(x + y) \leq w(x) + w(y)$,
- (4) if $t_n \rightarrow t$, $w(x_n - x) \rightarrow 0$, then $w(t_n x_n - tx) \rightarrow 0$.

Every metrizable linear topology is given by a paranorm which is total ($w(x) = 0$ implies $x = 0$) and strictly increasing ($0 < t < 1$, $x \neq 0$ implies $w(tx) < w(x)$). If w is such a paranorm and $U = \{x | w(x - x_0) < \epsilon\}$, then U is shrinkable at x_0 in the sense of Ives [2] and Klee [4]. That is, $t \in [0, 1)$, $x \in \bar{U}$ implies $x_0 + t(x - x_0) \in U$.

We state the main result of the paper.

(1) **Theorem.** *Suppose X is a linear topological space and $A \subset X$. If X admits a weak, incomplete, total, strictly increasing paranorm w such that A is closed in the completion of w , then A is negligible.*

The theorem applies to the familiar spaces of functions on $[0, 1]$ (including L^p , $0 < p < 1$), since these, except for the space of all measurable functions, admit the weak, incomplete paranorm $\int_{[0,1]} |f|/(1 + |f|)$. A similar ap-

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plication can be made to sequence spaces, using the paranorm $w(x) = \sum |x_n|/2^n(1 + |x_n|)$.

While we cannot use the distance from A to construct a homeomorphism on the whole space (as Bessaga does), we can, using distance, define homeomorphisms on certain closed subspaces and extend linearly. If P is a closed hyperplane in a normed space and $x \notin P$, then the translates $P_n = P + nx$ divide the space, and projection in direction x into P_n is continuous. A similar division may be effected in a general linear topological space by substituting for P the boundary of a cone over a shrinkable set.

(2) **Theorem.** *Suppose U is an open subset of a linear topological space Y and that U is shrinkable at $x_0 \neq 0$. Suppose also $0 \notin \bar{U}$ and $C = \{\lambda x | x \in U, \lambda > 0\}$. Then C is open, and if $y \in \bar{C}$, $\mu > 0$, then $y + \mu x_0 \in C$. Each line $\{z + \lambda x_0 | \lambda \in (-\infty, \infty)\}$ intersects $\text{Bd } C$ exactly once.*

Proof. It is clear that C is open. For each $c \in C$, choose $\lambda(c) > 0$, $u(c) \in U$ so that $c = \lambda(c)u(c)$. Suppose λ is unbounded on each neighborhood of y . For each neighborhood W of y and each positive integer n , choose $c_{w,n}$ such that $c_{w,n} \in W \cap C$ and $\lambda(c_{w,n}) > n$. $u(c_{w,n}) = (1/\lambda(c_{w,n}))c_{w,n} \rightarrow 0$, since $c_{w,n} \rightarrow y$. Thus $0 \in \bar{U}$, a contradiction. Now let W be a neighborhood of y on which λ is bounded, and suppose $y_\delta \rightarrow y$ with $y_\delta \in W \cap C$. If $\lambda(y_\delta) \rightarrow 0$, take τ such that $0 < \tau < \mu$. Since $(1/\lambda(y_\delta))y_\delta \in U$,

$$\lambda(y_\delta)\tau^{-1}((1/\lambda(y_\delta))y_\delta - x_0) \in U - x_0 \quad \text{for } \delta > \delta_0,$$

since U is shrinkable at x_0 . Thus $\tau^{-1}y \in \bar{U} - x_0$ so that $\mu^{-1}y = (\tau/\mu)\tau^{-1}y \in U - x_0$ and $y + \mu x_0 \in \mu U \subset C$. Otherwise $\lambda(y_\delta)$ has a convergent subnet $\lambda(y_{\delta(\alpha)}) \rightarrow \lambda_0 > 0$.

$$u(y_{\delta(\alpha)}) = (1/\lambda(y_{\delta(\alpha)}))y_{\delta(\alpha)} \rightarrow \lambda_0^{-1}y \in \bar{U}.$$

Thus

$$(\lambda_0/(\lambda_0 + \mu))(\lambda_0^{-1}y - x_0) \in U - x_0,$$

from which it follows that $y + \mu x_0 \in (\lambda_0 + \mu)U \subset C$. It remains to show that a line $\{z + \lambda x_0\}$ meets $\text{Bd } C$ exactly once. It is now sufficient to show that the line meets both C and the complement of C . If for each n , $z + nx_0 \notin C$, then $x_0 + n^{-1}z = n^{-1}(z + nx_0) \notin C$ and $x_0 \notin C$, a contradiction. If for every positive integer n , $z - nx_0 \in C$, then $n^{-1}z - x_0 \in C$, so that $-x_0 \in \bar{C}$. By the first part of the theorem this implies $0 \in C$.

(3) **Theorem.** *If C is as in (2) and $q(y) = \{y + \lambda x_0 \mid \lambda \in (-\infty, \infty)\} \cap \text{Bd } C$ for each $y \in Y$, then q is continuous.*

Proof. For $x \in Y$, let $\lambda(x)$ be such that $q(x) = x + \lambda(x)x_0$. Suppose that on each neighborhood of y , λ is not bounded above. For each neighborhood W of y and each positive integer n , choose $y_{w,n} \in W$ such that $\lambda(y_{w,n}) > n$. Then $y_{w,n} + nx_0 \notin C$, so that $n^{-1}y_{w,n} + x_0 \notin C$. This implies $x_0 \notin C$, a contradiction. We can similarly show λ is bounded below on some neighborhood of y . Let W be a neighborhood on which λ is bounded and $y_\delta \rightarrow y$ with $y_\delta \in W$. Now suppose $\lambda(y_{\delta(\alpha)})$ is a subnet of $\lambda(y_\delta)$ converging to λ_0 . Then $q(y_{\delta(\alpha)}) \rightarrow y + \lambda_0 x_0$. Since $\text{Bd } C$ is closed, $y + \lambda_0 x_0 \in \text{Bd } C$. Since $y + \lambda(y)x_0$ is also in $\text{Bd } C$, $\lambda_0 = \lambda(y)$. That is, every convergent subnet of $\{\lambda(y_\delta)\}$ converges to $\lambda(y)$. Thus $\lambda(y_\delta) \rightarrow \lambda(y)$, and $q(y_\delta) \rightarrow q(y)$.

(4) **Lemma.** *Suppose X is a Hausdorff linear topological space and Y a closed subset of X . If $h: Y \rightarrow X$ is continuous and 1-1, and there is a compact set $K \subset X$ such that $h(y) - y \in K$ for each $y \in Y$, then h is a topological embedding. If K is also a linear set and $Y = X$, then h is a homeomorphism onto X .*

Proof. The continuity of h^{-1} needs to be verified. For $x \in h(Y)$, $x = h(y) = y + k$ with $k \in K$, so that $y = h^{-1}(x) = x - k$. Suppose $x_\delta, x \in h(Y)$ and $x_\delta \rightarrow x$. $h^{-1}(x_\delta) = x_\delta - k_\delta$ with $k_\delta \in K$. If $k_{\delta(\alpha)} \rightarrow k$ is a convergent subnet, then $h^{-1}(x_{\delta(\alpha)}) = x_{\delta(\alpha)} - k_{\delta(\alpha)} \rightarrow x - k \in Y$. Thus $x_{\delta(\alpha)} = h h^{-1}(x_{\delta(\alpha)}) \rightarrow h(x - k)$, so that $x = h(x - k)$ and $h^{-1}(x) = x - k$. This shows every convergent subnet of $\{k_\delta\}$ converges to k , and so $h^{-1}(x_\delta) = x_\delta - k_\delta \rightarrow x - k = h^{-1}(x)$. If $Y = X$, and K is linear, then the intermediate value theorem applies to each line in direction K to show h is onto X .

We proceed to the proof of (1). Let Y be the w -completion of X and $e \in Y \setminus X$. It is not hard to find a sequence $\{z_n\} \subset X$ such that $z_n \rightarrow e$, $z_1 = 0$ and $w(z_{n+1} - z_n) < w(z_n - z_{n-1})/10$ for each $n \geq 2$. Let $x_n = z_{n+1} - z_n$. We will define a sequence of homeomorphisms whose composition will be the desired mapping. The first will be in direction x_1 and move points y in a neighborhood of A to $y + x_1$. Points far from A will not be moved. We first define the homeomorphism on translates of the boundary of a cone and then fill in linearly. There exists λ such that $w(x_1) < w(\lambda x_1) < 2w(x_1)$. With C as the cone of (2) over $\{x \in Y \mid w(x - x_1) < w(x_1)/2\}$, let

$$P_n = (\text{Bd } C) + n\lambda x_1 \quad \text{and} \quad D = \bigcup \{P_n \mid n \text{ an integer}\}.$$

Let

$$w_1(y) = \min\{w(y), w(x_1)\} \quad \text{and} \quad A_1 = \{y \in Y \mid d(y, A) \leq 5w(x_1)\}.$$

For $y \in D$, we take

$$h_1(y) = y + (w(x_1) - d_1(y, A_1))(1/w(x_1))x_1.$$

Here $d_1(y, A_1) = \inf\{w_1(y - z) \mid z \in A_1\}$. h_1 is extended by requiring that it map $[y, y + \lambda x_1]$ linearly onto $[h_1(y), h_1(y + \lambda x_1)]$ for each $y \in D$. Using (2), it is easy to see that h_1 is thereby defined on all of Y . The mapping $h_1|D$ is clearly 1-1 and continuous, and using the fact that projection onto P_n is continuous, these properties carry to the extension. By (4), h_1 is a homeomorphism onto Y . Notice that $\{x \in X \mid w(x - x_1) < w(x_1)/2\}$ is open and shrinkable in X with respect to the original topology, and that the boundary of $C \cap X$ in X is $(\text{Bd}_w C) \cap X$. Since the distance d_1 is also continuous in the given topology, our argument proves that $h_1|X$ is a homeomorphism of X onto itself.

If $y \in [y', y'']$ with $y' \in P_n$, $y'' = y' + \lambda x_1$, then $h_1(y) = y$ if $h_1(y') = y'$, $h_1(y'') = y''$; and $h_1(y) = y + x_1$ if $h_1(y') = y' + x_1$, $h_1(y'') = y'' + x_1$. Using these facts it is not hard to show $d(y, A) \leq 10w(x_2)$ implies $h_1(y) = y + x_1$, and $d(y, A) \geq 10w(x_1)$ implies $h_1(y) = y$. h_2 will be defined in the same way as h_1 , moving points in direction x_2 , fixing those outside a $10w(x_2)$ neighborhood of $A + x_1$, and translating a $10w(x_3)$ neighborhood V of $A + x_1$ to $V + x_2$. Homeomorphisms $h_3, h_4, \dots$ are similarly defined, and since each point in $X \setminus A$ has a neighborhood on which $\{h_n \dots h_2 h_1\}$ is eventually fixed, we may define a homeomorphism $h: X \setminus A \rightarrow X$ by $h = \dots h_3 h_2 h_1$. It is not hard to see that the range of h is all of X .

An isotopy theorem such as Klee's in [3] may also be stated in this context. To see this let $h_0 = \text{Id}$, and define for $x \in X$, $t \in [1 - 2^{-n}, 1 - 2^{-n-1}]$, $f_t(x) = \phi(j_t(x))$, where

$$\phi(x) = x + \frac{t - 1 + 2^{-n}}{2^{-n} - 2^{-n-1}} (h_{n+1}(x) - x) \quad \text{and} \quad j_t(x) = h_n \dots h_2 h_1 h_0(x).$$

ϕ is 1-1, since h_{n+1} is monotone increasing on each line in direction x_{n+1} . By (4), ϕ is a homeomorphism onto X , so that f_t is also. Notice that $f_0 = \text{Id}$. Letting $f_1(x) = h(x)$ for each $x \in X \setminus A$ and $H(x, t) = (f_t(x), t)$, we check that H is an invertible isotopy. H maps $X \times [1 - 2^{-n}, 1 - 2^{-n-1}]$ onto itself in a continuous 1-1 fashion. $H(x, t) - (x, t) = (f_t(x) - x, 0)$ lies in the compact subset

$$([0, z_2] \cup [z_2, z_3] \cup \dots \cup [z_{n+1}, z_{n+2}]) \times \{0\}$$

of $X \times R$, so (4) shows $H|X \times [1 - 2^{-n}, 1 - 2^{-n-1}]$ is a homeomorphism. We thus have $H|X \times [0, 1)$ is a homeomorphism of $X \times [0, 1)$ onto itself, and it is a routine matter to check the continuity of H at points $(x, 1)$, with $x \in X \setminus A$, and of H^{-1} at points $(x, 1)$, $x \in X$.

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DEPARTMENT OF MATHEMATICS, KEENE STATE COLLEGE, KEENE, NEW HAMPSHIRE 03431