GENERATION OF ABSTRACT-PARAMETER SEMIGROUPS

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ABSTRACT. This paper generalizes the Hille-Yosida-Phillips theorem for abstract-parameter semigroups.

1. Introduction. Generation is a fundamental problem in one-parameter operator semigroup theory and it has been done by many authors. (See, for instance, [3], [1] and the references they contain.) It is, in fact, introducing and characterising an operator, the infinitesimal generator, which determines the structure of the semigroup. In 1944, E. Hille introduced a study of operator semigroups with abstract parameter [3, pp. 327–336 and Chapter XXV]. This has since been studied by Tulcea [8], Langlands [5], [6], Ivanova [4], Chabrowski and Lubzonok [2] and others. However, the notion of the infinitesimal generator for abstract-parameter semigroups, in general, does not seem feasible making it difficult to discuss its generation as above. The main purpose of this paper is to discuss the generation of this kind of semigroup under a suitable generalized notion of generation. Specifically, it involves replacing the “infinitesimal generator” naturally with the cone of infinitesimal generators associated with an abstract-parameter semigroup and characterising cones which give rise to such semigroups. This, indeed, is a generalization of the notion of generation above, since for the one-parameter case characterising the infinitesimal generator or the cone of infinitesimal generators is equivalent.

2. Preliminaries. Let X, Y be complex or real B -spaces. Let X^+ be an open positive cone in X (that is, X^+ is an open subset of X such that if $x, x' \in X^+$ and $\alpha > 0$, then $x + x' \in X^+$ and $\alpha x \in X^+$). Let X^+ have the origin of X as a limit point. Let $S = \{S(x): x \in X^+\}$ be a semigroup of bounded linear operators on Y . Thus $S(x + x')y = S(x)S(x')y \quad \forall x, x' \in X^+, y \in Y$. S

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is called a partially C_0 semigroup on Y iff $\lim_{t \downarrow 0} S(tx)y = y$ for each $x \in X^+$ and $y \in Y$. It is called a (totally) C_0 semigroup on Y iff $\lim_{\|x\| \rightarrow 0} S(x)y = y$ for each $y \in Y$. It is clear that if X^+ is finite dimensional then a partially C_0 semigroup is C_0 . However, C_0 semigroups are partially C_0 .

Since X^+ is locally bounded, a C_0 semigroup S is strongly continuous on X^+ (cf. [8, pp. 59–60]). Moreover, it still retains the exponential bound well known in the case $X^+ = (0, \infty)$.

Theorem 1. *Let the semigroup S be C_0 . Then there exist positive constants σ and M such that*

$$\|S(x)\| \leq Me^{\|x\|\sigma} \quad \forall x \in X^+.$$

Further, if $B \subset X^+$ is bounded then there exists $\sigma_B \geq 0$ such that

$$\|S(tx)\| \leq Me^{t\sigma_B} \quad \forall t > 0, x \in B.$$

Proof. For $\delta > 0$, let $N(\delta) = \{x \in X^+ : \|x\| \leq \delta\}$. A familiar uniform boundedness argument shows that there are constants $\delta > 0$ and $M = M(\delta) \geq 1$ such that $\|S(x)\| \leq M \quad \forall x \in N(\delta)$. Now there is $\nu_0 > 0$ such that $\nu N(1) \subset N(\delta) \quad \forall 0 < \nu \leq \nu_0$. Thus $\|S(\nu x)\| \leq M \quad \forall x \in N(1), 0 < \nu \leq \nu_0$. Now given any $x \in X^+$, there is $0 < \tau \leq \nu_0$ and a nonnegative integer m such that $\|x\| = m\nu_0 + \tau$. Hence using a modification of a technique of Miyadera,

$$\begin{aligned} \|S(x)\| &= \left\| S\left(\|x\| \frac{x}{\|x\|}\right) \right\| = \left\| S\left(m\nu_0 + \tau\right) \frac{x}{\|x\|} \right\| \\ &\leq \left\| S\left(m\nu_0 \frac{x}{\|x\|}\right) \right\| \left\| S\left(\tau \frac{x}{\|x\|}\right) \right\| \leq \left\| S\left(\nu_0 \frac{x}{\|x\|}\right) \right\|^m \left\| S\left(\tau \frac{x}{\|x\|}\right) \right\| \\ &\leq M^{1+m} = MM^{(\|x\| - \tau)/\nu_0} \leq MM^{\|x\|/\nu_0} = Me^{\|x\|\sigma} \end{aligned}$$

($\sigma = \nu_0^{-1} \log M \geq 0$) and the first part of the theorem is proved. The second part follows immediately.

3. Generator, infinitesimal generators and resolvents. It is convenient for our purpose to see a cone of infinitesimal generators associated with S as an operator in $X^+ \times Y$ into Y . We designate such an operator ‘generator’. Thus the generator is S is defined by

$$A(x, y) = \lim_{t \downarrow 0} \frac{S(tx)y - y}{t}, \quad (x, y) \in X^+ \times Y,$$

and the domain $D(A)$ of A is the subset of $X^+ \times Y$ on which the limit exists. We establish some properties of the generator A and the resolvents $R(\lambda; A(x))$ of the $A(x)$'s, λ being in the resolvent set $\rho(A(x))$ of $A(x)$.

Theorem 2. *Let S be C_0 with A its generator. Then*

- (a) *A is linear in the second argument and positive linear in the first;*
- (b) *A is closed.*

Proof. For any fixed $x \in X^+$, $\{S(tx): t > 0\}$ is a one-parameter C_0 semigroup in Y , and $A(x) = A(x, \cdot)$ is its infinitesimal generator. The first part of (a) is therefore immediate and the second part follows as a consequence of [3, Theorem 10.8.5]. Consider (b). We adapt the classical technique for the infinitesimal generator. Let (x_n, y_n) in $D(A)$ be such that $(x_n, y_n) \rightarrow (x, y)$ in $X^+ \times Y$ and $A(x_n, y_n) \rightarrow z$ in Y . Now $\int_0^a S(tx_n)A(x_n)y_n dt = [S(ax_n) - I]y_n$. The boundedness of a convergent sequence, Theorem 1, the strong continuity of $S(x)$ and a $2 - \epsilon$ argument show that $\lim_{n \rightarrow \infty} S(tx_n)A(x_n)y_n = S(tx)z$ and $\|S(tx_n)A(x_n)y_n\| \leq Ke^{t\sigma}$ for some K and σ positive and all $t > 0$. Hence by the dominated convergence theorem and for similar reasons as above

$$\lim_{n \rightarrow \infty} \int_0^a S(tx_n)A(x_n)y_n dt = \int_0^a S(tx)z dt = \lim_{n \rightarrow \infty} [S(ax_n) - I]y_n = [S(ax) - I]y.$$

And, from the C_0 property of S ,

$$z = \lim_{a \downarrow 0} \frac{1}{a} \int_0^a S(tx)z dt = \lim_{a \downarrow 0} \frac{1}{a} [S(ax) - I]y.$$

Thus $(x, y) \in D(A)$ and $A(x, y) = A(x)y = z$. This completes the proof.

Theorem 3. *Let S be C_0 . Then*

- (1) *for $x, x' \in X^+$ and $\lambda > \sigma_{xx'} = \max(\sigma_x, \sigma_{x'})$,*

$$R(\lambda; A(x))R(\lambda; A(x')) = R(\lambda; A(x'))R(\lambda, A(x));$$

- (2) *if $B \subset X^+$ is bounded, then there exist positive constants M and σ_B such that*

$$\| [R(\lambda; A(x))]^n \| \leq M(\lambda - \sigma_B)^{-n} \quad \forall x \in B, \lambda > \sigma_B, n = 1, 2, \dots$$

Proof. (1) follows from the integral representation of $R(\lambda; A(x))$, the commutativity of the $S(x)$'s and the boundedness of these linear operators. (2) similarly follows.

4. Generation. Finally, we consider the problem: What property should an operator A possess to be the generator for S ?

Theorem 4. An operator A in $X^+ \times Y$ into Y is the generator of a unique C_0 semigroup S iff

- (1) A is additive in the first argument, linear in the second and closed;
- (2) for each $x \in X^+$, $D(A(x, \cdot))$ is dense in Y ;
- (3) for any bounded set $B \subset X^+$ there exist positive constants M_B and σ_B such that

$$\| [R(\lambda; A(x))]^n \| \leq M_B (\lambda - \sigma_B)^{-n} \quad \forall x \in B, \lambda > \sigma_B, y \in Y;$$

- (4) for any $x, x' \in X^+$ and any $\lambda \in \rho(A(x)) \cap \rho(A(x'))$,

$$R(\lambda; A(x))R(\lambda; A(x')) = R(\lambda; A(x'))R(\lambda; A(x)).$$

Furthermore,

$$S(x)y = \lim_{\lambda \rightarrow \infty} \exp(-\lambda) \sum_{n=0}^{\infty} \frac{\lambda^n [\lambda R(\lambda; A(x))]^n}{n!} y$$

for each $y \in Y$ and $x \in X^+$.

Proof. The "only if" part is clear. Consider the "if" part. A is closed in $X^+ \times Y$ and linear in the second argument. Hence $A(x, \cdot) = A(x)$ is a closed linear operator in Y . Along with hypotheses (2) and (3) it is well known [3, pp. 360–362] that for $x \in X^+$, $y \in Y$ and $\lambda > \sigma_x$

$$S_\lambda(tx)y \stackrel{\text{def}}{=} S_\lambda(t; A(x))y \stackrel{\text{def}}{=} \exp(-\lambda t) \sum_{n=0}^{\infty} \frac{(\lambda t)^n [\lambda R(\lambda; A(x))]^n}{n!} y$$

converges uniformly for t in any bounded subset of $[0, \infty)$. Hypothesis (3) makes the limit

$$S(tx)y \stackrel{\text{def}}{=} S(t; A(x))y \stackrel{\text{def}}{=} \lim_{\lambda \rightarrow \infty} S_\lambda(tx)y$$

converge uniformly for t in any bounded subset of $[0, \infty)$ and x in B . Then by a $3 - \epsilon$ argument, $\lim_{t \downarrow 0} S(tx)y = y$ uniformly w.r.t. x in B . And, therefore,

$$\lim_{\|x\| \rightarrow 0} S(x)y = \lim_{\|x\| \rightarrow 0} S\left(\|x\| \frac{x}{\|x\|}\right)y = y \quad (y \in Y).$$

We show next that $S(x)$ possesses the semigroup property in X^+ . Let $x, x' \in X^+$ be fixed and let z be any one of x, x' and $x + x'$.

(a) Then hypothesis (3) and the series representation of $S_\lambda(tx)$ give $\|S_\lambda(tz)\| \leq Me^{t\sigma}$ for some $\sigma > 0$ and all $t > 0, \lambda > \sigma$.

(b) Since by hypothesis (4) the resolvents commute, then $R(\lambda; A(z))$ commutes with $S_\lambda(tz) \forall \lambda > \sigma$ and $t \geq 0$.

(c) $dS_\lambda(tz)y/dt = S_\lambda(tz)\lambda A(z)R(\lambda; A(z))y \forall \lambda > \sigma, t > 0$ and $y \in Y$.

(d) $\lim_{\lambda \rightarrow \infty} \lambda R(\lambda; A(z))y = y$.

(See, for instance, [8] for (c) and (d).) It follows that for $\lambda > \sigma$ and $y \in D(A(x)) \cap D(A(x')) \cap D(A(x+x')) \stackrel{\text{def}}{=} D$,

$$\begin{aligned} & \|S_\lambda(x+x')y - S_\lambda(x)S_\lambda(x')y\| \\ &= \left\| \int_0^1 \frac{d}{dt} (S_\lambda((1-t)(x+x'))S_\lambda(tx)S_\lambda(tx')y) dt \right\| \\ &= \left\| \int_0^1 S_\lambda((1-t)(x+x'))S_\lambda(tx)S_\lambda(tx') \right. \\ &\quad \cdot [\lambda A(x)R(\lambda; A(x)) + \lambda A(x')R(\lambda; A(x')) - \lambda A(x+x')R(\lambda; A(x+x'))]y dt \left. \right\| \\ &\leq \int_0^1 M^3 e^{\sigma(1-t)} e^{2t\sigma} \|\lambda R(\lambda; A(x))A(x) + \lambda R(\lambda; A(x'))A(x') \\ &\quad - \lambda R(\lambda; A(x+x'))A(x+x')\|y\| dt \\ &= \sigma^{-1} M^3 e^{\sigma} (e^{\sigma} - 1) \|\lambda R(\lambda; A(x))A(x) - A(x) + \lambda R(\lambda; A(x'))A(x') \\ &\quad - A(x') + A(x+x') - \lambda R(\lambda; A(x+x'))A(x+x')\|y\| \end{aligned}$$

as $\lambda \rightarrow \infty$ for all $y \in D$; the last equality holding because A is additive in the first argument (hypothesis (1)). It is a consequence of hypothesis (2) that D is dense in Y . Since (a) also holds, it follows by the Banach-Steinhaus theorem that $\|S_\lambda(x+x')y - S_\lambda(x)S_\lambda(x')y\| \rightarrow 0$ as $\lambda \rightarrow \infty$ for all $y \in Y$. Hence $S(x+x')y = S(x)S(x')y$ for all $x, x' \in X^+, y \in Y$. Thus $\{S(x), x \in X^+\}$ is a C_0 semigroup in Y .

The rest of the proof is routine.

Theorem 5. *An operator A in $X^+ \times Y$ into Y is the generator of a partially C_0 semigroup S iff A is additive in the first argument; for each $x \in X^+, A(x)$ is the infinitesimal generator of a C_0 semigroup on Y ; and the resolvents of the $A(x)$'s commute.*

Proof. Similarly as above.

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