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Ext (A, T) AS A MODULE OVER End (T)

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ABSTRACT. In this paper we show that for abelian groups A and T, where A is of finite rank and T is torsion, the End (T)-module Ext (A, T) is finitely generated or is of finite rank.

1. **Definition.** Let M be a left R-module. We shall say that M has rank at most n if and only if every finitely generated submodule of M is contained in one generated by n elements. It has rank n if, in addition, n is the least integer for which this holds. In this case we write $rM = n$.

We remark that our definition of rank is the one used by Prüfer in his fundamental work [7] on torsion groups. This definition is given again in [4, p. 49]. It is equivalent to the usual definition for torsion-free and p-groups, and to that of reduced rank as given in [1, p. 34].

We note that this notion of rank has the following properties:

1. If M is a finitely generated abelian group $(R = Z)$, rM coincides with the number of elements in a canonical basis.

2. If M is a torsion-free abelian group, rM is the dimension of $M \otimes_Z Q$ over the field Q of rational numbers.

3. For any R, the rank of a homomorphic image of M does not exceed that of M.

4. For R-modules M_i, if $rM_i \leq k_i$, then $r(\bigoplus_{i=1}^n M_i) \leq \sum_{i=1}^n k_i$.

5. Abelian groups of finite rank are characterized by the following:

Theorem 1.1. (i) *A is of rank at most n (over Z) if and only if it is a subquotient of the direct sum of n groups each isomorphic to Q.*

(ii) *A is of rank at most n if and only if it is embeddable in a divisible group of rank n.*

Proof. (i) follows from (ii) which is easy.

Note that it is possible for a group of finite rank to be an infinite direct sum of nonzero groups.

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In what follows, the abelian groups A and T will be considered as left modules over Z, whereas $\mathrm{Ext}(A, T)$ will be considered as a left module over $\mathrm{End}(T)$, the ring of endomorphisms of T (see [5, p. 143]). As the proofs in §2 show, the results of this section apply equally well to $\mathrm{Pext}(A, B)$ in place of $\mathrm{Ext}(A, B)$, (see [3]).

When considering $\mathrm{Ext}(A, T)$ as a left module over $\mathrm{End}(T)$, the question arises whether it is finitely generated. The following lemma shows that even in special situations this is not the case.

Lemma 1.2. *Let T be a direct sum of cyclic p-groups of unbounded order; then* $\mathrm{Ext}(Q, T)$ *is not a finitely generated* $\mathrm{End}(T)$*-module.*

Proof. Suppose $E_1, \cdots, E_n$ generate $\mathrm{Ext}(Q, T)$, and let X_i be the middle groups of these extensions. By [6], [8] we know that there exist extensions $E\colon 0 \to T \to X \to Q \to 0$ where the p-Ulm sequence of X is smaller than those of the X_i for all i. One such X is given as follows. If the pointwise minimum of the equivalence classes representing the p-Ulm sequences of X_i is equivalent to the sequence $\{\omega, \omega + 1, \omega + 2, \cdots\}$, it is easy to see that such an X exists with its p-Ulm sequence finite since T is a direct sum of cyclic groups of unbounded order. However, if the minimum of the p-Ulm sequences of X_i is a sequence of integers, one constructs the desired X as in [8]. Now if $E = \Sigma_{i=1}^n \alpha_i E_i$, then the p-Ulm sequences of E and $\Sigma_{i=1}^n \alpha_i E_i$ would be equivalent. However it is not hard to see that both the module action of $\mathrm{End}(T)$, as well as the addition of extensions, do not decrease the p-Ulm sequence. Thus the equality $E = \Sigma_{i=1}^n \alpha_i E_i$ contradicts the choice of E and therefore $E_1, \cdots, E_n$ do not generate $\mathrm{Ext}(Q, T)$ as claimed.

Although, in general, $\mathrm{Ext}(A, T)$ is not finitely generated, we will show that locally it is finitely generated. Moreover we shall give a sufficient condition for $\mathrm{Ext}(A, T)$ to be finitely generated. We prove the following:

Theorem 1.3. $r\,\mathrm{Ext}(A, T) \leq 1$ *whenever A is any countable group and T is an unbounded reduced primary group.*

Theorem 1.4. *If T is torsion, then* $r\,\mathrm{Ext}(A, T) \leq rA$, *for any A with rA finite.*

Before stating the next theorem, we remark that when T is a torsion group of unbounded order (Theorem 1.3 and Theorem 1.5 with infinitely many primary components nonzero), $\mathrm{Ext}(Q, T)$ is (additively) an uncountable direct sum of rational groups.

Theorem 1.5. *If T is torsion, with each primary component of it being bounded, and if A is any abelian group with $rA \leq n$, then* Ext(A, T) *is generated as an* End(T)*-module by n elements.*

Whenever these modules turn out to be finitely generated we exhibit the generators.

2. **Proofs of the theorems.** For the notions of basic subgroup, final rank, etc. consult [2].

Theorem 1.3 is a special case of the following:

Theorem 2.1. *For torsion groups T, $r\mathrm{Ext}(A, T) \leq 1$ whenever the final rank of B_p, a basic subgroup of the p-primary component T_p, is $\geq |A|$ for all primes p with $T_p \neq 0$, where $|A|$ is the cardinal number of A.*

Proof. The End(T)-module Ext(A, T) is a quotient of a module of the form Hom(F, T), with $rF \leq |A|$, and F free. Here $rF = \dim_Q F \otimes_Z Q$, as is consistent with our definition when the rank of F is finite. The result now follows from Remark 3 and

Lemma 2.2. *Let T be a torsion group and F a free group. If $rF \leq$ final rank of B_p for all p with $T_p \neq 0$ then the rank of* Hom(F, T) *over* End(T) *is at most one.*

Proof. We shall give a proof in the case T is p-primary, as the general case is a modification of this proof. We have to show that given $\alpha, \beta \in$ Hom(F, T) there exists a $\gamma \in$ Hom(F, T) and $t, s \in$ End(T) such that $s\gamma = \alpha$, $t\gamma = \beta$.

Let S be a basis for F. We construct a countable number of subsets S_i, $i = 1, 2, 3, \cdots$, of T as follows. For each $x \in S$ if $\max(o(\alpha(x)), o(\beta(x))) = p^n$, put an element of order p^n in S_n. Observe that $|S_i| \leq rF$. Let B be basic in T and write $B = \bigoplus_{i=1}^{\infty} B_i$, where $\operatorname{fin} r(B_i) = \operatorname{fin} r(B)$. Since $\operatorname{fin} r(B_i) = \operatorname{fin} r(B) \geq rF$ there exists an injection σ_i from S_i to the set of generators of B_i of order $\geq p^i$.

We are now ready to define the homomorphisms s, t, and γ. Let $x \in S$. As above x determines an element of S_n for some n which in turn determines an element $b \in B_n$ under σ_n. Now define $s_1, t_1 \colon B \to T$ by $s_1(b) = \alpha(x)$, $t_1(b) = \beta(x)$. Send the remaining generators of B to zero. By a theorem of Szele, see [1, p. 106], there is a surjection $\omega \colon T \to B$. Let $\gamma \colon F \to T$ be defined by $\gamma(x) \in \omega^{-1}(b)$. Then clearly $s = s_1\omega$ and $t = t_1\omega$ are as required.

Alternatively one can replace T in Lemma 2.2 by $G \oplus F_1$ where G is arbitrary and F_1 free with $rF_1 \geq rF$.

Proofs of Theorems 1.4 and 1.5. We may assume that T is reduced. Let $rA = n$. By Theorem 1.1, A may be embedded in a divisible group D of the same rank n, say $D = D_1 \oplus \cdots \oplus D_n$, each D_i having rank 1. (Note that D_i need not be indecomposable.) Since the $\mathrm{End}(T)$-homomorphism $\mathrm{Ext}(D, T) \to \mathrm{Ext}(A, T)$ is onto, it is sufficient by Remark 3 to demonstrate the theorem for D in place of A. Moreover by the additivity of Ext and Remark 4, it suffices to give the proof for D of rank one.

Case I. $D = \bigoplus_p Z(p^\infty)$ *where p ranges over different primes.* Then

$$\mathrm{Ext}(D, T) = \mathrm{Ext}\left(\bigoplus_p Z(p^\infty), T\right) = \prod_p \mathrm{Ext}(Z(p^\infty), T) = \prod_p \mathrm{Ext}(Z(p^\infty), T_p),$$

T_p being the p-primary component of T. If T_p is bounded, then the isomorphism $\mathrm{Ext}(Z(p^\infty), T_p) \simeq T_p$ is an $\mathrm{End}(T_p)$ isomorphism. Since $\mathrm{End}(T)$ acts on $\mathrm{Ext}(Z(p^\infty), T_p)$ via the natural projection $\mathrm{End}(T) \to \mathrm{End}(T_p)$, and since in this case T_p is clearly cyclic over $\mathrm{End}(T_p)$, it follows that $\mathrm{Ext}(Z(p^\infty), T_p)$ is cyclic over $\mathrm{End}(T)$. If T_p is unbounded, then $r\,\mathrm{Ext}(Z(p^\infty), T_p) \leq 1$ by Theorem 1.3. The conclusion now follows from the following two facts:

1. Each $\mathrm{Ext}(Z(p^\infty), T_p)$ is cyclic or of rank ≤ 1 over $\mathrm{End}(T_p)$, depending on whether T_p is bounded or not.

2. Each T_p is fully invariant in T so that $\mathrm{End}(T) = \Pi_p \mathrm{End}(T_p)$ acts componentwise on $\Pi_p \mathrm{Ext}(Z(p^\infty), T_p)$.

Case II. $D = Q$. Consider the injective resolution of Z of the form $0 \to Z \to Q \to Q/Z \to 0$. Thus we have an exact sequence

$$\mathrm{Ext}(Q/Z, T) \to \mathrm{Ext}(Q, T) \to \mathrm{Ext}(Z, T) = 0.$$

Since Q/Z is torsion, divisible and $rQ/Z = 1$, it follows by Case I that $\mathrm{Ext}(Q/Z, T)$ is cyclic or of rank ≤ 1, depending on whether all the p-primary components of T are bounded or not. Thus the same holds for $\mathrm{Ext}(Q, T)$, this being an $\mathrm{End}(T)$-homomorphic image of $\mathrm{Ext}(Q/Z, T)$. This completes the proofs of Theorems 1.4 and 1.5.

We give two examples. The first shows that one cannot strengthen the conclusion of Theorem 1.3 from locally cyclic to cyclic. In addition to the examples provided by Lemma 1.2, when $T = \bigoplus_{i=1}^\infty Z(p^i)$ and $A = \bigoplus_{\aleph_0} Z(p)$ (or $A = Z(p) \oplus Z(p)$ or $A = Z(p)$), then $\mathrm{Ext}(A, T)$ is not cyclic over $\mathrm{End}(T)$.

Moreover when $A = Z(p) \oplus Z(p)$, $T = Z(p^n)$, $n \in N$, then $r\mathrm{Ext}(A, T) = 2$ and, thus, Theorem 1.4 holds with equality rather than inequality.

We now produce generators for Ext(A, T) as stated in Theorem 1.5. From the proof we see that we only need to do this for two cases: (I) $A = Z(p^\infty)$ and T a bounded p-group, and (II) $A = Q$ and $T = \bigoplus_p T_p$ where T_p are bounded p-groups for all p.

Case I. $A = Z(p^\infty)$ *and* T *a bounded p-group*. Consider the exact sequence Γ: $0 \to Z \xrightarrow{i} Q_p \xrightarrow{\eta} Z(p^\infty) \to 0$, where $Q_p = \{n/p^i \mid n, i \in Z\}$. Let p^k be the bound on T. Then $T = T_1 \oplus Z(p^k)$. Consider $E_0 \in \mathrm{Ext}(Z(p^\infty), T)$,

$$E_0 \colon 0 \to T_1 \oplus Z(p^k) \xrightarrow{\hat{i}} T_1 \oplus Q_p/\langle p^k \rangle \xrightarrow{\hat{\eta}} Z(p^\infty) \to 0,$$

where $\hat{i}|_{T_1} = I$ and $\hat{i}(0, 1) = (0, \bar{1})$, where $\bar{1}$ denotes the coset of 1 in $Q_p/\langle p^k \rangle$ and $\hat{\eta}$ is the projection. We claim E_0 is a generator. Let $E \in \mathrm{Ext}(Z(p^\infty), T)$. We need to produce an $\alpha \in \mathrm{End}(T)$ such that $\alpha E_0 = E$. Let E: $0 \to T_1 \oplus Z(p^k) \xrightarrow{\chi} X \xrightarrow{\sigma} Z(p^\infty) \to 0$. Consider the diagram

$$\begin{tikzcd}
 & & & \Gamma \\
 & & & 0 \arrow[d] & \\
 & & & Z \arrow[d, "i"] \arrow[dll, dashed, "\exists!\bar{\alpha}"'] & \\
E' \colon 0 \arrow[r] & T_1 \oplus Z(p^k) \arrow[r] \arrow[d, equal] & T_1 \oplus Z(p^k) \oplus Q_p \arrow[r, shift left, "\pi_2"] \arrow[d, "\phi", "\simeq"'] & Q_p \arrow[r] \arrow[l, shift left, "i_2"] \arrow[d, equal] & 0 \\
E\zeta \colon 0 \arrow[r] & T_1 \oplus Z(p^k) \arrow[r] \arrow[d, equal] & N \arrow[r, "p_2"] \arrow[d, "p_1"] & Q_p \arrow[r] \arrow[d, "\eta"] & 0 \\
E \colon 0 \arrow[r] & T_1 \oplus Z(p^k) \arrow[r, "\chi"] & X \arrow[r, "\sigma"] & Z(p^\infty) \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

where $N = \{(x, r) \mid \sigma(x) = \eta(r)\}$ and $E\eta \equiv E'$ is the split extension since $T_1 \oplus Z(p^k)$ is a pure bounded subgroup of N. It is easy to see that $\sigma p_1 \phi i_2 = \eta$, so $\sigma p_1 \phi i_2 i = 0$ which implies there exists a unique $\bar{\alpha}\colon Z \to T_1 \oplus Z(p^k)$ such that $\chi\bar{\alpha} = p_1 \phi i_2 i$. This implies $E = \bar{\alpha}\Gamma$. Set $\gamma = p_1 \phi i_2$.

Observe that $\langle p^k \rangle \subset$ kernel γ, so we get $\bar{\gamma}: Q_p/\langle p^k \rangle \to X$. Moreover $\bar{\alpha}: Z \to T_1 \oplus Z(p^k)$ induces $\alpha \in \operatorname{End}(T)$ defined by $\alpha|_{T_1} = 0$ and $\alpha(0, 1) = \bar{\alpha}(1)$. Now the diagram

$$\begin{tikzcd}
E_0: 0 \arrow[r] & T_1 \oplus Z(p^k) \arrow[r, "\hat{i}"] \arrow[d, "\alpha"'] & T_1 \oplus Q_p/\langle p^k \rangle \arrow[r, "\hat{\eta}"] \arrow[d, "\bar{\gamma}"] & Z(p^\infty) \arrow[r] \arrow[d, equal] & 0 \\
E\ : 0 \arrow[r] & T_1 \oplus Z(p^k) \arrow[r, "\chi"] & X \arrow[r, "\sigma"] & Z(p^\infty) \arrow[r] & 0
\end{tikzcd}$$

commutes where we set $\bar{\gamma}|_{T_1} = 0$. This implies $E = \alpha E_0$ as desired.

Case II. $A = Q$, $T = \bigoplus_p T_p$ *where* T_p *are bounded* p*-groups*. Let P be the set of all primes. Write $P = \Delta_1 \cup \Delta_2$, where $\Delta_1/\{p \in P | T_p \neq 0\}$ and $\Delta_2 = P \setminus \Delta_1$. For $p_i \in \Delta_1$, let $p_i^{k_i}$ be the bound on T_{p_i}. Define a group G by generators and relations. Let the generators be g, g_{ij} over all primes $p_i \in P$ and $j \in N$ and the relations $p_i^{k_i} g - p_i g_{i1}$, $p_i g_{i,j+1} - g_{ij}$ for all $p_i \in \Delta_1$ and $j \in N$; $g - p_i g_{i1}$, $p_i g_{i,j+1} - g_{ij}$ for all $p_i \in \Delta_2$ and $j \in N$. Observe that the torsion subgroup of G, $T(G) = \bigoplus_{p_i \in \Delta_1} Z(p_i^{k_i})$ and $G/T(G) \simeq Q$. Since T_{p_i} is bounded by $p_i^{k_i}$ so $T_{p_i} = T'_{p_i} \oplus Z(p_i^{k_i})$. Hence E_0: $0 \to T \xrightarrow{\mu} (\bigoplus_{p_i \in \Delta_1} T'_{p_i}) \oplus G \xrightarrow{\pi} Q \to 0$ is exact. We now show that E_0 is a generator. Let E: $0 \to T \xrightarrow{\chi} X \xrightarrow{\sigma} Q \to 0$ be exact. Choose $x \in X$ such that $\sigma(x) = 1$. Since $X/T \simeq Q$ and $p_i^{k_i} T_{p_i} = 0$ there exist elements $x_{ij} \in X$ such that $p_i^{k_i} x = p_i x_{i1}$, $p_i x_{i,j+1} = x_{ij}$ for all $p_i \in \Delta_1$ and $j \in N$. Moreover for all $p_i \in \Delta_2$ and $j \in N$ there exist elements $x_{ij} \in X$ such that $x = p_i x_{i1}$, and $p_i x_{i,j+1} = x_{ij}$. Observe that the homomorphism $f: (\bigoplus_{p_i \in \Delta_1} T'_{p_i}) \oplus G \to X$ defined by $f|_{\oplus_{p_i \in \Delta_1} T'_{p_i}} = 0$, $f(g) = x$, $f(g_{ij}) = x_{ij}$ makes the diagram

$$\begin{tikzcd}
E_0: 0 \arrow[r] & T \arrow[r, "\mu"] \arrow[d, dashed, "\exists!\, \hat{f}"'] & \left(\bigoplus_{p_i \in \Delta_1} T'_{p_i} \right) \oplus G \arrow[r, "\pi"] \arrow[d, "f"] & Q \arrow[r] \arrow[d, equal] & 0 \\
E\ : 0 \arrow[r] & T \arrow[r, "\chi"] & X \arrow[r, "\sigma"] & Q \arrow[r] & 0
\end{tikzcd}$$

commute. This implies $\hat{f} E_0 = E$ as desired.

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