SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

NORMALIZING ORTHOGONAL POLYNOMIALS BY USING THEIR RECURRENCE COEFFICIENTS

ALAN G. LAW AND M. B. SLEDD

ABSTRACT. Polynomials $P_n(x)$ generated by a three-term recurrence relation with suitably restricted coefficients are orthogonal with respect to a (perhaps unknown) distribution $da(x)$ on the real line. An evaluation of $\int_a^b P_n^2(x) da(x)$ is given in terms of the coefficients in the recurrence relation. Knowledge of the distribution $da(x)$ is unnecessary.

A three-term recurrence

$$(1) \quad \begin{aligned} P_0(x) &= 1, \quad P_1(x) = A_0x + B_0, \\ P_{n+1}(x) &= (A_nx + B_n)P_n(x) - C_nP_{n-1}(x), \quad n \geq 1, \end{aligned}$$

where A_n ($n \geq 0$), B_n ($n \geq 0$), and C_n ($n \geq 1$) are real numbers for which $A_n C_{n+1} \neq 0$ ($n \geq 0$), generates a sequence $\{P_n\}$ of polynomials in which P_n is of degree exactly n . A necessary and sufficient condition that the polynomials P_n be orthogonal on an interval $[a, b]$ of the real line is [1], [3] that

$$(2) \quad C_n / (A_n A_{n-1}) > 0 \quad (n \geq 1).$$

Remark. Orthogonality, as used here, means that there exists an integrator $\alpha(x)$ over an interval $[a, b]$ ($-\infty \leq a < b \leq +\infty$) such that

$$\int_a^b P_i(x) P_j(x) d\alpha(x) = 0, \quad i \neq j,$$

where $\alpha(x)$ is bounded, is nondecreasing, and assumes infinitely many different values over $[a, b]$.

Received by the editors July 8, 1974.

AMS (MOS) subject classifications (1970). Primary 33A65; Secondary 42A52.

Key words and phrases. Orthogonal polynomials, three-term recurrence relation, normalization.

Copyright © 1975, American Mathematical Society

If (2) is satisfied, the norm of each P_n can be computed directly from the coefficients in (1), as indicated by the following

Theorem. Suppose in (1) that $C_n/(A_n A_{n-1}) > 0$ for all $n \geq 1$. Then

$$\int_a^b P_n^2(x) d\alpha(x) = \frac{A_0}{A_n} C_1 C_2 C_3 \cdots C_n \int_a^b d\alpha(x), \quad n \geq 1.$$

Proof. Since

$$x^n = (A_0 A_1 A_2 \cdots A_{n-1})^{-1} P_n(x) + \sum_{j=0}^{n-1} a_j P_j(x)$$

for some coefficients $a_0, a_1, a_2, \dots, a_{n-1}$,

$$\int_a^b x^n P_n(x) d\alpha(x) = (A_0 A_1 \cdots A_{n-1})^{-1} \int_a^b P_n^2(x) d\alpha(x), \quad n \geq 1.$$

Multiplication throughout the recurrence (1) by x^{n-1} , followed by integration and use of the orthogonality, then yields that

$$\int_a^b P_n^2(x) d\alpha(x) = \frac{C_n A_{n-1}}{A_n} \int_a^b P_{n-1}^2(x) d\alpha(x), \quad n \geq 1.$$

The proof can now be completed with a simple induction argument.

Remark. Though the evaluation given by the theorem for $\int_a^b P_n^2(x) d\alpha(x)$ depends on $\int_a^b d\alpha(x)$, it is customary to normalize the integrator $\alpha(x)$ in some way (for example, by requiring that $\int_a^b d\alpha(x) = 1$); and then the dependence of the evaluation on the integrator disappears entirely.

This theorem has an obvious analog for recursively generated polynomials which possess an orthogonality different from the type described above. As an illustration, the Bessel polynomials P_n satisfy a recurrence (1) in which $C_n/(A_n A_{n-1}) < 0$ for $n \geq 1$; hence [3] they are not orthogonal polynomials associated with some distribution over some interval. However, they are orthogonal polynomials [2] with weight $w(x) = \exp(-2/x)$ around the unit circle U in the complex plane. A direct analog of the preceding theorem implies that the Bessel polynomials P_n satisfy

$$\int_U P_n^2(x) \exp(-2/x) dx = \frac{(-1)^n}{2n+1} \int_U \exp(-2/x) dx = \frac{(-1)^{n+1} 4\pi i}{2n+1}$$

for $n \geq 0$.

REFERENCES

1. J. Favard, *Sur les polynomes de Tchebicheff*, C. R. Acad. Sci. Paris 200 (1935), 2052-2053.

2. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc. 65 (1949), 100–115. MR 10, 453.

3. A. G. Law, *Solutions of some countable systems of ordinary differential equations*, Doctoral Dissertation, Georgia Institute of Technology, 1968.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SASKATCHEWAN, REGINA,
SASKATCHEWAN, CANADA

DEPARTMENT OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY,
ATLANTA, GEORGIA 30332