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WEAK CHAINABILITY OF PSEUDOCONES

DAVID P. BELLAMY

ABSTRACT. A pseudocone over X is a compactification of $(0, 1]$ with remainder X. S is a circle. A characterization of those pseudocones over S which are weakly chainable is given. (A continuum is weakly chainable if and only if it is a continuous image of the pseudoarc.) Covering projections and liftings are used, and a simple geometric interpretation of the result is that a pseudocone over S is weakly chainable if and only if the absolute value of the winding number of any subarc of $(0, 1]$ around S is bounded by some $m > 0$.

The following terminology will be used here. A continuum is a compact connected metric space. $I = [0, 1]$; $A = (0, 1]$; S is the unit circle in the complex numbers. If X is a continuum, a *pseudocone over* X is a compactification of A with remainder X [1], [2]. R is the set of real numbers and Z is the set of integers. A continuum X is *acyclic* if every continuous map f: $X \to S$ is nullhomotopic.

A. Lelek has introduced the notion of a weakly chainable continuum in [4]. The importance of these continua is that they are precisely the continuous images of the pseudoarc [4], [5]. In conversation, Sam B. Nadler, Jr. and J. Quinn have raised the question of which pseudocones over S are weakly chainable. This is related to a problem posed by them in [6, Remark 6.2, p. 67]. It is the purpose of this note to provide an answer to this question. This is also of interest since Lelek's examples in [4, p. 28] are both pseudocones over circles. See also [3].

The definition of weak chainability is rather technical and is available in [4]. It is really unrelated to the present work, so it will not be given here. Instead, we use the following

Lemma. *A continuum is weakly chainable if and only if it is a continuous image of a chainable continuum.*

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Proof. This is immediate from the Theorem in [4, p. 274] and Corollaries 1 and 3 of [4, p. 276].

Now, let p: $\mathbf{R} \to S$ be the usual covering projection; $p(t) = e^{it}$. Let P be a pseudocone over S, where $S \subseteq P$ and j: $A \to P$ is the embedding of A. Let r: $P \to S$ be a retraction. (One always exists by Proposition 1 of [1, p. 7] or Theorem 3 of [1, p. 19].) Let h: $P \to S \times I$ be the embedding given by: $h(s) = (s, 0)$, $s \in S$; $h(j(s)) = (r \circ j(s), s)$, $s \in A$. Let $p \times 1$: $\mathbf{R} \times I \to S \times I$ be defined in the usual way. Then $p \times 1$ is a covering projection also. Let $Q = (p \times 1)^{-1}(h(P))$. Let $\{J_k\}_{k \in Z}$ be the set of all liftings of $r \circ j$ through p, where $0 \leq J_0(1) < 2\pi$, and where $J_{k+1}(t) = J_k(t) + 2\pi$ for each t, k.

Let K_k: $A \to R \times I$ be given by $K_k(t) = (J_k(t), t)$. Then $Q = R \times \{0\} \cup [\bigcup_{k \in Z} K_k(A)]$. Now observe that the image $J_k(A)$ is a bounded subset of R if and only if $K_k(A)$ is a bounded subset of $R \times I$ in the Euclidean metric. Also, if $J_k(A)$ is bounded for any k, it is for every k.

Theorem. *Let P be a pseudocone over S. Using the notation in the preceding discussion, P is weakly chainable if and only if $J_0(A)$ is a bounded subset of R.*

Proof. Suppose $J_0(A)$ is bounded. Let $X = \mathrm{Cl}(K_0(A))$. Then X is a pseudocone over a closed interval, and so is chainable by Lemma 2.7 of [2, p. 298]. Further, $(p \times 1)(\mathrm{Cl}\ K_0(A)) = h(P)$, so that, by the Lemma, $h(P)$ is weakly chainable. Thus P is also weakly chainable.

Conversely, suppose P is weakly chainable, let X be a chainable continuum and let f: $X \to P$ be a continuous surjection. By the proof of Lemma 1 of [7, pp. 74, 75], there is a lifting F: $X \to \mathbf{R} \times I$ of $h \circ f$ through $p \times 1$. Then $F(X) \subseteq Q$, since $(p \times 1)(F(X)) = h(P)$.

Since $K_k(1) \in F(X)$ for some k, we may choose F so that $K_0(1) \in F(X)$. Then $K_0(t) \in F(X)$ for all $t \in A$; for if $K_0(t) \notin F(X)$ for some $t < 1$, then $M = \{K_0(t)\colon\ t > x\}$ is a nonempty proper closed and open subset of $F(X)$, contradicting the connectedness of X. ($M \neq \emptyset$, since $K_0(1) \in M$; $M \neq F(X)$, since $F(X) \cap (R \times \{0\}) \neq \emptyset$, while $M \cap (R \times \{0\}) = \emptyset$. M is open and closed in $F(X)$ since

$$M = \{(J_0(t) + s, t)\colon -\epsilon < s < \epsilon;\ 1 \geq t > x\} \cap F(X)$$

$$= \{(J_0(t) + s, t)\colon -\epsilon \leq s \leq \epsilon;\ 1 \geq t \geq x\} \cap F(X)$$

where $0 < \epsilon < 2\pi$.)

Thus, we must have $K_0(A) \subseteq F(X)$. Since $F(X)$ is compact, it is bounded in $R \times I$. Thus $K_0(A)$ is bounded and, hence, so is $J_0(A)$.

Corollary. *A pseudocone P over S is weakly chainable if and only if it is a continuous image of some acyclic continuum.*

Proof. If P is weakly chainable it is a continuous image of a chainable continuum which is acyclic. If F: $X \longrightarrow P$ is a continuous surjection, where X is acyclic, then, again, by the proof of Lemma 1 of [7, pp. 74, 75], there is a lifting of F of $h \circ f$ through $p \times 1$ (using the above notation). Again, without loss of generality, $K_0(A) \subset F(X)$ and, hence, $K_0(A)$ is bounded. Thus, $J_0(A)$ is bounded and P is weakly chainable.

There is a simple geometric interpretation of the above Theorem. P is weakly chainable if and only if there exists an $m > 0$ such that for any $s, t \in A$, where $s < t$, the absolute value of the "winding number" of $h \circ j | [s, t]$ is strictly less than m. (Intuitively, it is clear what "winding number" means here—the net algebraic number of times which $h \circ j$ wraps $[s, t]$ around S in $S \times I$. To define "winding number" precisely here, we shall have to return to the lifting J_0 and define the winding number to be, say, $(J_0(s) - J_0(t))/2\pi$. It would suffice to consider only those s, t for which this is an integer.)

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19711