L^1_X IS WEAKLY COMPACTLY GENERATED IF X IS

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ABSTRACT. Though good criteria for weak compactness in the space of Bochner-integrable functions are not yet known, one can show that $L^1(\mu; X)$ is a weakly compactly generated Banach space whenever μ is finite and X is a weakly compactly generated Banach space. The proof depends upon a recent factorization scheme due to W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński.

Let (Ω, Σ, μ) denote a finite positive measure space and X a Banach space. Let $L^1(\mu; X)$ be the space of μ -Bochner-integrable X -valued functions defined on Ω ; the usual almost-everywhere identifications are assumed.

It is a basic open problem in the theory of vector measures to determine necessary and sufficient conditions for a bounded subset K of $L^1(\mu; X)$ to be relatively weakly compact. In case X is a reflexive Banach space, then a bounded subset K of $L^1(\mu; X)$ is relatively weakly compact if and only if

$$\lim_{\mu(A) \rightarrow 0} \int_A \|k(\omega)\| d\mu(\omega) = 0$$

uniformly for $k \in K$; this is a result of S. Chatterji [3]. Slightly more general results have been obtained by J. Batt [1], J. K. Brooks and N. Dinculeanu [2], and C. Swartz [10]. All these results assume something on the order of both X and X^* possessing the so-called Radon-Nikodým property. Thus the question of general criteria seems far from being settled.

We do, however, have one result in this direction.

Theorem. *If X is a weakly compact generated Banach space, then so is $L^1(\mu; X)$.*

Proof. First, we observe that if Y is any Banach space and $T: Y \rightarrow X$ is a continuous, linear operator with TY dense in X , then the Σ -simple TY -valued functions are dense in $L^1(\mu; X)$. This is an easy consequence of the density of Σ -simple functions in $L^1(\mu; X)$ and the fact

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that if $\epsilon > 0$ is given and

$$s = \sum_{i=1}^n c_{A_i} \otimes x_i$$

is a Σ -simple, X -valued function, then choosing y_i such that

$$\|Ty_i - x_i\| \leq \frac{\epsilon}{\mu(A_i)n}$$

yields

$$\left\| s - \sum_{i=1}^n c_{A_i} \otimes Ty_i \right\|_1 \leq \epsilon.$$

Next, we note that if X is weakly compactly generated then there exists a reflexive Banach space Y and a continuous linear operator $T: Y \rightarrow X$ such that TY is dense in X : this is just Corollary 3 of [4].

From the above two paragraphs we conclude that if X is weakly compactly generated, then there exists a reflexive Banach space Y and a continuous linear operator $S: L_1(\mu; Y) \rightarrow L_1(\mu; X)$ such that the range of S is dense in $L_1(\mu; X)$. Indeed, S is defined by $Sf = T \circ f$ for all $f \in L_1(\mu; Y)$, where T is as given in the preceding paragraph.

Now $L_1(\mu; Y)$ is weakly compactly generated; this is obvious from Chatterji's result: $\{c_A \otimes y: A \in \Sigma, \|y\| \leq 1\}$ is a relatively weakly compact subset of $L_1(\mu; Y)$ having dense linear span. We then conclude that $L_1(\mu; X)$ is also weakly compactly generated, since S maps dense sets to dense sets.

Remark. The factorization scheme of Davis et al. can be applied in many other ways to problems related to vector measures; see, for example, the Epilogue of [6]. This is due to the important fact (as is Chatterji's result) that reflexive Banach spaces possess the Radon-Nikodým property.

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