THE DEGREE OF PIECEWISE MONOTONE INTERPOLATION

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ABSTRACT. Let $0 = x_0 < x_1 < \cdots < x_k = 1$ and let $y_0, y_1, \dots, y_k$ be real numbers such that $y_{j-1} \neq y_j$, $j = 1, 2, \dots, k$. Estimates are obtained on the degree of an algebraic polynomial $p(x)$ that interpolates the given data piecewise monotonely; i.e., such that (i) $p(x_j) = y_j$, $j = 0, 1, \dots, k$, and such that (ii) $p(x)$ is increasing on $I_j = (x_{j-1}, x_j)$ if $y_j < y_{j-1}$, and decreasing on I_j if $y_j > y_{j-1}$, $j = 1, 2, \dots, k$. The problem is seen to be related to the problem of monotone approximation.

Let $0 = x_0 < x_1 < \cdots < x_k = 1$ and let $y_0, y_1, \dots, y_k$ be real numbers such that $y_{j-1} \neq y_j$, $j = 1, 2, \dots, k$. It is a result of Wolibner [7], Kammerer [2], and Young [8] that there exists an algebraic polynomial $p(x)$ such that:

- (i) $p(x_j) = y_j$, $j = 0, 1, \dots, k$, and
- (ii) $p(x)$ is increasing on $I_j = (x_{j-1}, x_j)$ if $y_j > y_{j-1}$, and decreasing on I_j if $y_j < y_{j-1}$, $j = 1, 2, \dots, k$.

A polynomial $p(x)$ with properties (i) and (ii) is said to interpolate *piecewise monotonely*; in case $y_j > y_{j-1}$ for all j (or if $y_j < y_{j-1}$ for all j), $p(x)$ is simply said to interpolate *monotonely*. The smallest degree of a polynomial that interpolates the values $Y = \{y_0, y_1, \dots, y_k\}$ at the points $X = \{x_0, x_1, \dots, x_k\}$ (piecewise) monotonely is called the *degree of (piecewise) monotone interpolation of Y with respect to X* , and is denoted by $N = N(X; Y)$. Rubinstein has obtained estimates for the degree of monotone interpolation for the special case $k = 2$ [6]. We seek general estimates on $N(X; Y)$. Let

$$\Delta = \Delta(Y) = \min_{1 \leq j \leq k} |y_j - y_{j-1}|, \quad \text{and} \quad M = M(X; Y) = \max_{1 \leq j \leq k} \left| \frac{y_j - y_{j-1}}{x_j - x_{j-1}} \right|.$$

The estimates on $N(X; Y)$ are found (not too surprisingly) to be related

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to the degree of monotone and comonotone approximation. Let $f \in C[0, 1]$, with a finite number of relative extrema $0 = x_0 < x_1 < x_2 < \cdots < x_{j+1} = 1$ (such a function is called a *piecewise monotone function*). The relative extrema $x_1, x_2, \dots, x_j$ are called the *peaks* of f . The *degree of comonotone approximation* of f by algebraic polynomials of degree $\leq n$ is defined by

$$E_n^*(f) = \inf\{\|f - p\| : p \in P(n; f)\} \quad (\text{sup norm on } [0, 1]),$$

where $P(n; f)$ is the set of algebraic polynomials of degree $\leq n$, monotone on each of the subintervals (x_{i-1}, x_i) , $i = 1, 2, \dots, j+1$, with the same monotonicity as f on these intervals. If f is monotone on $[0, 1]$, then $E_n^*(f)$ is called the *degree of monotone approximation* of f .

Let S_j , $j = 0, 1, 2, \dots$, be the set of all piecewise monotone functions f with j peaks such that

$$(1) \quad \sup_{0 \leq x, y \leq 1} \left| \frac{f(x) - f(y)}{x - y} \right| \leq 1.$$

The *degree of comonotone approximation* to S_j is defined by

$$E_n^*(S_j) = \sup\{E_n^*(f) : f \in S_j\}.$$

It is known that $\lim_{n \rightarrow \infty} E_n^*(S_j) = 0$ [3], [4]. The smallest degree n such that $E_n^*(S_j) < \delta$ will be denoted by $n_j(\delta)$.

With given data X and Y we associate a piecewise linear function $L(x) = L(X; Y; x)$ defined by $L(x_i) = y_i$, $i = 0, 1, \dots, k$. Let j be the number of peaks of L . We state our main results:

Theorem 1. $N(X; Y) \leq n_j(\Delta/12M)$.

Theorem 2. If $y_0 < y_1 < \cdots < y_k$, then there exists an absolute constant A such that $N(X; Y) \leq AM/\Delta$.

We will first prove Theorem 1. Theorem 2 will follow from Theorem 1 and estimates on the degree of monotone approximation. The proof is based on an idea of Kammerer [2, Theorem 4.1], later used by Ford and Roulier [1].

Proof of Theorem 1. Let $\epsilon = \frac{1}{4}\Delta$, and let S be the set of 2^{k+1} piecewise linear functions f such that $f(x_i) = y_i + \epsilon$ or $y_i - \epsilon$, $i = 0, 1, \dots, k$. We enumerate the functions in S and denote the i th function in S by f_i . Note that our choice of ϵ guarantees that each f_i is comonotone with $L(x)$ and that $|f_i(x) - f_i(y)| \leq 3M|x - y|/2$ for all $x, y \in [0, 1]$. Thus $2f_i/3M \in S_j$, so that there exists $p_i \in P[n_j(\Delta/12M); L]$ such that $\|2f_i/3M - p_i\| < \Delta/12M$. Let $q_i = 3Mp_i/2$. Then $q_i \in P[n_j(\Delta/12M); L]$ and $\|f_i - q_i\| < (3M/2)(\Delta/12M) = \Delta/8 = \epsilon/2$.

The vector $(y_0, y_1, \dots, y_k)$ is thus contained in the convex hull of the vectors $(q_i(x_0), q_i(x_1), \dots, q_i(x_k))$, $i = 1, 2, \dots, 2^{k+1}$, and therefore there exists a convex linear combination of the q_i 's which will give rise to a polynomial p which interpolates Y piecewise monotonely. Since the degree of each $q_i \leq n_j(\Delta/12M)$, we have $N(X; Y) \leq n_j(\Delta/12M)$.

Lemma (Lorentz and Zeller [3, Theorem 2]). *There exists a constant c such that $E_n^*(S_0) \leq c/n$.*

Proof of Theorem 2. From the Lemma we obtain $n_0(\Delta/12M) \leq 12c(M/\Delta)$. It follows from this and Theorem 1 that $N(X; Y) \leq AM/\Delta$.

While Theorem 2 may fail to give the exact value of $N(X; Y)$ for a particular configuration (e.g., if all the points lie on a straight line), it is best possible in a classwise sense, as we shall show in Theorem 3.

Let $G(M; \Delta)$ be the set of all $(X; Y)$ such that $M(X; Y) \leq M$ and $\Delta(Y) \geq \Delta$. Let $N(G) = \sup\{N(X; Y) : (X; Y) \in G\}$.

Theorem 3. *Let $y_0 < y_1 < \dots < y_k$. Then there exist constants $c_1, c_2 > 0$ such that*

$$(2) \quad c_1 M/\Delta \leq N(G) \leq c_2 M/\Delta.$$

Proof. For each $(X; Y) \in G$, $N(X; Y) \leq AM(X; Y)/\Delta(Y)$, by Theorem 2. Thus the upper bound in (2) holds with $c_2 = A$.

For the lower bound, note that Theorem 2 was proved using estimates on $E_n^*(S_0)$ and that this theorem, in turn, may be used to give estimates on $E_n^*(S_0)$. Indeed, if $f \in C[0, 1]$ is monotone and satisfies (1), we may choose $X = \{0 = x_0, x_1, \dots, x_k = 1\}$ such that

$$(3) \quad 1/n \leq f(x_i) - f(x_{i-1}) \leq 2/n, \quad i = 1, 2, \dots, k.$$

Let $Y = \{f(x_0), f(x_1), \dots, f(x_k)\}$. Then $\Delta(Y) \geq 1/n$ and $M(X; Y) \leq 1$ (by (1)), so that, by Theorem 2, there exists a polynomial $p \in P(A_n; f)$ such that $p(x_i) = f(x_i)$, $i = 0, 1, \dots, k$. Since p is a monotone interpolation of Y and f satisfies (3), we have $\|f - p\| \leq 2/n$. Thus $E_{A_n}^*(f) \leq 2/n$, so that $E_n^*(S_0) \leq A_1/n$, which is the result of Lorentz and Zeller contained in the Lemma. If $N(G) = o(M/\Delta)$, it would follow that there exists a sequence of polynomials $q_n \in P(n; f)$ which would satisfy $\|f - q_n\| = o(1/n)$. Since the result of Lorentz and Zeller is essentially unimprovable, this is impossible. Thus, there exists $c_1 > 0$ such that $N(G) \geq c_1 M/\Delta$.

Remark. It follows from [4] and [5] that there exists $c_3 > 0$ such that

$$(4) \quad n_j(\Delta/12M) \leq \min_{P \geq j+2} [c_3 P^2 2^P M/\Delta]^{(P+1)/(P-j-1)}.$$

From this result we can obtain an estimate on $N(X; Y)$ in the general case of j peaks. We have, in particular, that for any $\epsilon > 0$ there exists a constant $A_{j,\epsilon}$ such that

$$(5) \quad N(X; Y) \leq A_{j,\epsilon} (M/\Delta)^{1+\epsilon}.$$

We believe, however, that this estimate is not the best possible one, since (4) is probably short of best possible. In fact, the proof of the lower bound (2) in Theorem 3 can be modified to show that in the general case of j peaks, $N(G) \geq c_4 M/\Delta$ for some constant c_4 . We therefore conjecture that there exists d_j such that $N(X; Y) \leq d_j M/\Delta$, where j is the number of peaks of $L(X; Y; x)$. It may even be possible to replace d_j by an absolute constant d .

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