THE SUPPORTS OF INFINITELY DIVISIBLE DISTRIBUTION FUNCTIONS

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ABSTRACT. An infinitely divisible probability measure over the real line whose associated Lévy spectral measure gives nonzero mass to every deleted neighborhood of the origin is shown to have as its support an interval of the form $(-\infty, a]$, $[a, \infty)$ or $(-\infty, \infty)$.

1. Introduction. This is the beginning of a study of the supports of infinitely divisible distribution functions. By the support of a distribution function F , $\text{supp } F$, we mean the smallest closed set S for which $\int_S dF(x) = 1$. In a recent paper [4] it was shown that if F is infinitely divisible, without Gaussian component and whose Lévy spectral measure is absolutely continuous with respect to Lebesgue measure and is infinite, then the support of F is an interval of the form $(-\infty, a]$, $[a, \infty)$ or $(-\infty, \infty)$, and that the density of F is positive almost everywhere over its support. In this note it is shown that a relaxed hypothesis on the Lévy spectral measure M , namely, $M((-\epsilon, 0) \cup (0, \epsilon)) > 0$ for all $\epsilon > 0$, yields the same conclusion about the support of F .

The representation that will be used here for the characteristic function $f(u)$ of an infinitely divisible distribution function $F(x)$ is given by

$$(1) \quad f(u) = \exp \left\{ i\gamma u - \sigma^2 u^2 / 2 + \int_{x \neq 0} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) M(dx) \right\},$$

where γ and $\sigma^2 \geq 0$ are constants and where $M(x)$, the Lévy spectral measure, is a nondecreasing function over $(-\infty, 0)$ and over $(0, \infty)$, satisfying $M(-\infty) = M(+\infty) = 0$ and $\int_{-1}^0 + \int_0^1 x^2 M(dx) < \infty$. In what follows, the same symbol will be used for a distribution function (or any nondecreasing function) and the Lebesgue-Stieltjes measure it determines, i.e., if G is a distribution function, then $G(x) = G((-\infty, x])$, and $G(A) = \int_A dG(x)$.

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2. **The theorem.** The sole purpose of this note and this section is to state and prove the following result.

Theorem. *If F is an infinitely divisible distribution function with characteristic function given by (1) such that $M((-\epsilon, 0) \cup (0, \epsilon)) > 0$ for every $\epsilon > 0$, then the support of F is an interval of the form $(-\infty, a]$, $[a, \infty)$ or $(-\infty, \infty)$.*

Proof. In case $\sigma^2 > 0$, then $\text{supp } F = (-\infty, \infty)$; we need therefore only prove the theorem in case $\sigma^2 = 0$. We prove it in three cases.

Case (i). In this case, it is assumed that F has a characteristic function of the form

$$(2) \quad f(u) = \exp \int_0^\infty (e^{iux} - 1)M(dx),$$

where by hypothesis $M((0, \epsilon)) > 0$ for all $\epsilon > 0$ and $\int_{+0}^1 x M(dx) < \infty$. By Theorem 1 in [5], if X is a random variable with distribution function F , then $\text{ess inf } X = 0$ and $\text{sup } X = \infty$. We show that if $S = \text{supp } F$, then $S = [0, \infty)$. Suppose not. Then S being closed implies $[0, \infty) \setminus S$ is open and nonempty. Let (a, b) be any nonempty open interval in $[0, \infty) \setminus S$. Since $M((0, \epsilon)) > 0$ for all $\epsilon > 0$, there exists a sequence $x_n \downarrow 0$ such that each x_n is a point of increase of M , i.e., for every $\delta > 0$, $M(x_n + \delta) - M(x_n - \delta) > 0$ for all n . Hence, there exists an integer N such that $0 < x_N < b - a$. Let k be a positive integer such that $a < kx_N < b$. Let $0 < \delta_0 < x_N/2$, and let F_1 and F_2 be two distribution functions whose corresponding characteristic functions are

$$f_1(u) = \exp \left\{ \int_{+0}^{x_N - \delta_0} + \int_{x_N + \delta_0}^\infty \right\} (e^{iux} - 1)M(dx)$$

and

$$f_2(u) = \exp \int_{x_N - \delta_0}^{x_N + \delta_0} (e^{iux} - 1)M(dx).$$

We now show that $F_2((a, b)) > 0$. Let us denote $\lambda = M(x_N + \delta_0) - M(x_N - \delta_0) > 0$, and define

$$\begin{aligned} G(x) &= 0 \quad \text{if } x < x_N - \delta_0, \\ &= (M(x) - M(x_N - \delta_0))/\lambda \quad \text{if } x_N - \delta_0 \leq x < x_N + \delta_0, \\ &= 1 \quad \text{if } x > x_N + \delta_0, \end{aligned}$$

where M and G are taken as right continuous. Let $\delta > 0$ be such that $a < k(x_N - \delta) < k(x_N + \delta) < b$. Now $G((x_N - \delta, x_N + \delta)) > 0$ since G has a

point of increase at x_N . Letting $\{X_1, \dots, X_k\}$ be independent, identically distributed random variables with common distribution function G , we have

$$\begin{aligned} 0 < (G(x_N + \delta) - G(x_N - \delta))^k &= \prod_{j=1}^k P[X_j \in (x_N - \delta, x_N + \delta)] \\ &= P\left(\bigcap_{j=1}^k [X_j \in (x_N - \delta, x_N + \delta)]\right) \leq P\left[\sum_{j=1}^k X_j \in (k(x_N - \delta), k(x_N + \delta))\right] \\ &= G^{*k}((k(x_N - \delta), k(x_N + \delta))) \leq G^{*k}((a, b)), \end{aligned}$$

thus proving that $G^{*k}((a, b)) > 0$, where G^{*k} denotes the k -fold convolution of G with itself. But, as is well known,

$$F_2((a, b)) = \sum_{n \geq 0} e^{-\lambda} \frac{\lambda^n}{n!} G^{*n}((a, b)) \geq e^{-\lambda} \frac{\lambda^k}{k!} G^{*k}((a, b)) > 0,$$

thus proving $F_2((a, b)) > 0$. Now let U_1 and U_2 be two independent random variables whose distribution functions are F_1 and F_2 respectively. Since $x_N - \delta_0 > 0$, then by Theorem 1 in [5], $\text{ess inf } U_1 = 0$. Let $0 < \delta_1 < \frac{1}{2} \max\{b - kx_N, kx_N - a\}$. Now $F_1([0, \delta_1)) > 0$. In proving $F_2((a, b)) > 0$ above, we actually proved

$$F_2((kx_N - \delta_1, kx_N + \delta_1)) > 0.$$

Thus,

$$\begin{aligned} 0 &< F_1([0, \delta_1)) F_2((kx_N - \delta_1, kx_N + \delta_1)) \\ &= P[U_1 \in [0, \delta_1)] P[U_2 \in (kx_N - \delta_1, kx_N + \delta_1)] \\ &= P([U_1 \in [0, \delta_1)] [U_2 \in (kx_N - \delta_1, kx_N + \delta_1)]) \\ &\leq P[U_1 + U_2 \in (kx_N - \delta_1, kx_N + 2\delta_1)] \\ &\leq P[U_1 + U_2 \in (a, b)] = F((a, b)). \end{aligned}$$

This contradicts the fact that $(a, b) \subset [0, \infty) \setminus S$, thus proving the theorem in Case (i).

Case (ii). Assume now that F has the characteristic function

$$(3) \quad f(u) = \exp \int_{+0}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) M(dx), \quad \text{where } \int_{+0}^1 xM(dx) = \infty.$$

We shall show in this case that $\text{supp } F = (-\infty, +\infty)$. As in [4], we define a

measure M_1 by

$$\begin{aligned} M_1((x, \infty)) &= M((x, \infty)) \quad \text{if } x \geq 1, \\ &= M((1, \infty)) + \int_x^1 tM(dt) \quad \text{if } 0 < x < 1. \end{aligned}$$

As was shown in [4], M_1 is a Lévy spectral measure such that $\int_{+0}^1 xM_1(dx) < \infty$ and such that $M_1((0, 1)) = \infty$, and hence M_1 satisfies the hypothesis of Case (i). Let $M_2 = M - M_1$. As in the proof of Theorem 2 in [4], M_2 is a Lévy spectral measure, and $\int_{+0}^1 xM_2(dx) = \infty$. Now let F_1 and F_2 be two infinitely divisible distribution functions whose corresponding characteristic functions are given by

$$f_1(u) = \exp \left\{ \int_0^\infty (e^{iux} - 1)M_1(dx) \right\}$$

and

$$f_2(u) = \exp \left\{ \int_0^1 \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) M_2(dx) - iu \int_1^\infty \frac{x}{1+x^2} M_1(dx) \right\}.$$

Then $F = F_1 * F_2$. Applying Case (i) to F_1 , we obtain $\text{supp } F_1 = [0, \infty)$. By Theorem 1 in [5], F_2 is the distribution function of a random variable which is unbounded in both directions. In order to show that $\text{supp } F = (-\infty, +\infty)$, it is sufficient to show that if $a < b$ are arbitrary, then $F((a, b)) > 0$. Since $F_2(x) > 0$ for all x , there exists a point of increase x_2 of F_2 such that if x_1 is defined by $x_1 = (a + b)/2 - x_2$, then $x_1 > 0$. From Case (i), x_1 is a point of increase of F_1 . Thus, by an easy argument, $x_1 + x_2 = (a + b)/2$ is a point of increase of $F_1 * F_2$, which implies $F((a, b)) > 0$, thus proving the theorem in Case (ii).

Case (iii). The general case. If

$$\int_{0 < |x| < 1} |x|M(dx) = \infty,$$

then by Case (ii), $\text{supp } F = (-\infty, +\infty)$. So we must consider the case where

$$\int_{0 < |x| < 1} |x|M(dx) < \infty.$$

Let F_0 , F_1 and F_2 be distribution functions whose corresponding characteristic functions are

$$f_0(u) = e^{iu\gamma}, \quad \text{where } \gamma = - \int_{-\infty}^\infty \frac{x}{1+x^2} M(dx),$$

$$f_1(u) = \int_{-\infty}^0 (e^{iux} - 1)M(dx), \quad f_2(u) = \int_0^\infty (e^{iux} - 1)M(dx).$$

If either $M((-\infty, 0)) = 0$ or $M((0, \infty)) = 0$, then by Case (i), we have the conclusion of the theorem, where

$$\text{supp } F = \text{supp } F_1 * F_0 = (-\infty, \gamma]$$

if $M((0, \infty)) = 0$, and

$$\text{supp } F = \text{supp } F_2 * F_0 = [\gamma, \infty)$$

if $M((-\infty, 0)) = 0$. If $M((-\infty, 0))M((0, \infty)) > 0$, then either $\text{supp } F_2 * F_0 = [\gamma, \infty)$ and $F_1(x) > 0$ for all x , which implies $\text{supp } F = (-\infty, \infty)$, or $\text{supp } F_1 * F_0 = (-\infty, \gamma]$ and $F_2(x) < 1$ for all x , which also implies $\text{supp } F = (-\infty, +\infty)$. Q.E.D.

From the proof given above, one can easily compute the value of the constant a in the conclusion of the theorem for the various cases that arise.

Corollary. *If F is a continuous infinitely divisible distribution function, then the support of F is always of the forms $(-\infty, \infty)$, $(-\infty, a]$ or $[a, \infty)$.*

Proof. By a theorem due to Blum and Rosenblatt [1], Hartman and Wintner [3] and Doeblin [2], F is continuous if and only if $\sigma^2 > 0$ or

$$M((-\epsilon, 0) \cup (0, \epsilon)) = \infty \quad \text{for all } \epsilon > 0.$$

Thus the hypothesis of the theorem above is satisfied, thus yielding the conclusion.

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