STEINITZ CLASSES OF CYCLIC EXTENSIONS OF DEGREE l^r

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ABSTRACT. Let k be an algebraic number field and l an odd prime. The set of Steinitz classes of tamely ramified cyclic extensions K/k of degree l^r is calculated and shown to be a subgroup of the ideal class group of k .

Introduction. In this paper k will denote a (finite) algebraic number field with ring of algebraic integers O_k . For a finite extension K/k the ring O_K of integers in K is a module over O_k . Such modules are classified up to isomorphism by two invariants: the rank and the Steinitz class. Since the ring O_K is uniquely determined within K it is reasonable to denote the Steinitz class of O_K as an O_k -module by $C(K, k)$.

Denote by G a cyclic group of odd prime power order l^r . As K varies over the normal extensions of k with Galois group isomorphic to G the associated Steinitz class $C(K, k)$ varies over a subset of the ideal class group $C(k)$ of k ; this subset is denoted $R(k, G)$. It is the set of ideal classes of k which can be realized as the Steinitz class of the ring of integers in a normal extension K/k with Galois group G . The subset $R_l(k, G)$ of $R(k, G)$ consisting of those classes which correspond to tamely ramified extensions is a subgroup of $C(k)$ which will be explicitly described. The reader may find [4] and [6] to be useful introductory references. In reference [4] the problem was studied for the special case $r = 1$. The methods of this paper are the same in broad outline. In order to carry them through it is necessary to be able to construct tamely ramified cyclic extensions K/k of degree l^r .

The paper is divided into three parts. In the first it is shown that $R_l(k, G)$ is contained in a certain subgroup of $C(k)$. In the second is given the procedure for constructing tamely ramified cyclic extensions of degree l^r , which is then applied in the third to prove that the inclusion shown in part one is in fact an equality.

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The entire investigation is founded on a formula for $C(K, k)$ due to Artin [1]. Following [4], [6] this will be used in the form: $C(K, k)$ is the class of the square root of the relative discriminant ideal when K/k is a normal extension of odd degree.

1. The following standard lemma will be useful:

Lemma. *Let K/k be an abelian extension of number fields which is tamely ramified at the prime $\mathfrak{p}$ of k . Then $N(\mathfrak{p}) \equiv 1 \pmod{e(\mathfrak{p}, K/k)}$. (N denotes the absolute norm; e the degree of ramification in the indicated extension.)*

Proof. Let D denote the decomposition group of $\mathfrak{p}$ for K/k and $K/\mathbb{F}$ the residue field of K at some prime $\mathbb{F}$ which divides $\mathfrak{p}$. Define $f: D \rightarrow (K/\mathbb{F})^*$ by $f(\sigma) = \sigma(\pi)/\pi \pmod{\mathbb{F}}$ where $\pi \in \mathbb{F} \setminus \mathbb{F}^2$ is a prime element for $\mathbb{F}$. The function f satisfies the identity $f(\sigma\tau) = f(\sigma)f(\tau)^\sigma$; the exponent σ on the right means the automorphism of $K/\mathbb{F}$ over $k/\mathfrak{p}$ induced from σ . Let T denote the inertia group of $\mathfrak{p}$ in K/k ; $\tau \in T$ iff τ acts trivially on $K/\mathbb{F}$. Since D is abelian one sees that $f(\tau)^\sigma = f(\tau) \forall \tau \in T$. As every automorphism of $K/\mathbb{F}$ over $k/\mathfrak{p}$ is induced by some element of D it follows that $f(T) \subseteq (k/\mathfrak{p})^*$. The restriction of f to T is a homomorphism so that $|T| \mid (N(\mathfrak{p}) - 1)$; since $e = |T|$, this completes the proof.

One could also observe that the lemma follows from local class field theory. An extension K/k will be called "of type G " if it is normal and its Galois group is isomorphic to G .

Proposition. *Let K/k be a tamely ramified extension of type G and let W be the subgroup of $C(k)$ generated by $\{C(\mathfrak{p}): \mathfrak{p} \text{ prime}, N(\mathfrak{p}) \equiv 1 \pmod{l^r}\}$. Then $C(K, k) \in W^{(l-1)/2}$.*

Proof. Let $\mathfrak{d}$ be the relative discriminant ideal so that $C(K, k) = C(\mathfrak{d}^{1/2})$ by Artin's theorem. If a prime $\mathfrak{p}$ of k splits into g primes in K each of degree f and ramification degree e , then the contribution of $\mathfrak{p}$ to $\mathfrak{d}$ is $\mathfrak{d}_{\mathfrak{p}} = \mathfrak{p}^{fg(e-1)}$. Suppose $e = l^t$, $t > 0$; then $fg = l^{r-t}$. The lemma implies that $N(\mathfrak{p}) \equiv 1 \pmod{l^t}$; hence $N(\mathfrak{p}^{l^{r-t}}) \equiv 1 \pmod{l^r}$. From this it is easy to conclude that $C(\mathfrak{p}^{l^{r-t}}) \in W$. In fact $W = \{C(\alpha): N(\alpha) \equiv 1 \pmod{l^r}\}$; for let W_0 be the indicated set. Then it is obvious from the definition of W that $W \subseteq W_0$. Conversely, suppose $N(\alpha) \equiv 1 \pmod{l^r}$. By the generalized "theorem on primes in an arithmetic progression" there is a prime ideal $\mathfrak{q}$ such that $\alpha \mathfrak{q}^{-1}$ is a principal ideal with a generator $\alpha \equiv 1 \pmod{l^r}$ (congruence in the ring O_k). It follows that

$C(\alpha) = C(q)$ and that $N(q) \equiv 1 \pmod{l^r}$, so that $C(\alpha) \in W$.

The proposition should now be clear since for every prime $p \nmid b$ we have seen that $C(\frac{b}{p}) \in W^{l-1}$. It follows from the proposition that $R_l(k, G) \subseteq W^{(l-1)/2}$.

2. In order to prove that $R_l(k, G)$ and $W^{(l-1)/2}$ are actually equal it is necessary to be able to construct extensions K/k of type G . This will be done by constructing Kummer extensions of the field $F = k(\zeta)$ where ζ is a primitive l^r th root of unity. In the case $r = 1$ which was studied in [4] an extension L/F of degree l which is abelian over k contains a unique subfield K which is cyclic over k ; when $r > 1$ the situation is less simple but there proves to be a degree of uniqueness in any case. The construction which will be given does not produce all possible extensions K/k of type G but does give enough for present needs.

In addition to the notation which has already been introduced is the following:

ζ : primitive l^r th root of unity,

$F = k(\zeta)$,

Z = cyclic group generated by ζ ,

$m = [F : k]$,

$H = \text{Gal}(F/k)$,

σ : a fixed generator of H ,

s : defined by $\sigma(\zeta) = \zeta^s$,

$R = \mathbb{Z}/l^r\mathbb{Z}$,

$k^* = k \setminus \{0\}$ as a group under multiplication.

For $a \in F^*$ let $a_i = \sigma^i(a)$ and α_i be an l^r th root of a_i in some algebraic closure of k . Let $N(a) = F(\alpha_0, \dots, \alpha_{m-1})$. The field $N(a)$ is unambiguously determined by a even though the α_i 's are defined only up to a factor ζ^j . When only one a is under consideration $N(a)$ will be written N . Let $\mathcal{G} = \text{Gal}(N/k)$, $X = \text{Gal}(N/F)$, and $E = \{\beta \in N^* : \beta^{l^r} \in F\}$. There is a nondegenerate bilinear function $\langle \cdot, \cdot \rangle : E/F^* \times X \rightarrow Z$ = the cyclic group generated by ζ in F^* defined by $\langle e, \tau \rangle = \tau(e)/e$.

Let σ_1 be a fixed element of $\mathcal{G}$ whose restriction to F is the fixed $\sigma \in H$. E/F^* is a left module over the group ring RH by the action $\sigma(eF^*) = \sigma_1(e)F^*$ which will be written $\sigma(e)F^*$ from now on. X is a right RH -module

by the action $x^\sigma = \sigma_1^{-1} x \sigma_1$; Z is a left RH -module for the Galois action of H on F . With these actions the isomorphism of abelian groups $\phi: E/F^* \rightarrow \text{Hom}(X, Z)$ given by $\phi(eF^*) = \langle e, \cdot \rangle$ is an RH -isomorphism. H acts on $\text{Hom}(X, Z)$ via the left action on Z and the right action on X so that $(\sigma f)(x) = \sigma(f(x^\sigma))$. This means that

$$\langle \sigma(e), x \rangle = \sigma(\langle e, x^\sigma \rangle) = \langle e, x^\sigma \rangle^s$$

so that $\langle e, x^\sigma \rangle = \langle \sigma(e), x \rangle^{s^{-1}} = \langle s^{-1} \sigma e, x \rangle$ where s^{-1} is to be understood as (a unit) in R . Observe that E/F^* and X are both cyclic RH -modules. (For example E/F^* is generated over RH by $\alpha_0 F^*$.)

The first task is to identify the maximal abelian extension of k in N . The following result is easy to prove and probably well known.

Lemma. *Let $0 \rightarrow X \xrightarrow{i} \mathfrak{G} \rightarrow H \rightarrow 1$ be an extension of an abelian group X by a cyclic group $H = \langle \sigma \rangle$. Then the commutator subgroup of $\mathfrak{G}$ is $i(X^{1-\sigma})$.*

From now on let $\alpha = \alpha_0$ which will be identified with $\alpha_0 F^*$ as an element of E/F^* . Any cyclic extension of F in N has the form $F((\sum r_i \sigma^i) \alpha)$ for some element $\sum r_i \sigma^i \in RH$. In view of the lemma such an extension is abelian over k iff

$$\left\langle \left(\sum r_i \sigma^i \right) \alpha, x^{1-\sigma} \right\rangle = 1 \quad \forall x \in X.$$

Since $\langle (\sum r_i \sigma^i) \alpha, x^{1-\sigma} \rangle = \langle (\sum r_i \sigma^i - \sum s^{-1} r_i \sigma^{i+1}) \alpha, x \rangle$, this will be the case iff the element $\sum (r_i - s^{-1} r_{i-1}) \sigma^i$ in RH annihilates α in E/F^* . (We agree that $r_{-1} = r_{m-1}$.) As this is certainly the case if each $r_i = rs^{-i}$ for some $r \in R$, the following is true.

Proposition. *Let $\epsilon = \sum s^{-i} \sigma^i \in RH$. Then for any $\beta \in E$, $F(\epsilon \beta)$ is an abelian extension of k .*

Remark. In the case $r = 1$ every cyclic extension L/F of degree l which is abelian over k can be obtained by adjoining $(\epsilon a)^{1/l}$ for some $a \in F^*$. The proof of this fact uses the semisimplicity of the group ring RH and the fact that ϵ is a primitive idempotent in RH . In the case being considered here RH is not semisimple and ϵ is not an idempotent; one would expect that in this case there are extensions L/F of type G which are abelian over k but which are not of the form $L = F((\epsilon a)^{1/l})$ for any $a \in F^*$. Here is an example of such an extension: Let $k = \mathbb{Q}$, $l^r = 9$, $F = \mathbb{Q}(\zeta)$ where ζ is a primitive 9th root of unity. L will be the field $F(\zeta^{1/9})$ which is certainly

abelian over Q . In terms of the action of RH on $F^*/F^{*l'}$ it must be shown that $\zeta \notin \epsilon a$ in $F^*/F^{*l'}$ for any a . (1) $O_F = \mathbb{Z}[\zeta]$ is a PID, for the relative class number is 1 (cf. Hasse [3]) and the class number of the maximal real subfield F^+ is also 1 (cf. tables in [2] or the review problem set III in Samuel [7]). (2) It is now easy to see that if $\zeta = \epsilon a$ then a may be taken as a unit in F . (3) Every unit in F has the form $\pm \zeta^i \eta$ where η is a unit in F^+ . (This follows from Satz 23 or 24 in [3].) (4) As 2 is a primitive root mod 9 we may take $\sigma(\zeta) = \zeta^2$. Then for any $\eta \in F^+$, $\epsilon \eta = \prod \sigma^i(\eta)^{s^{-i}}$, $s^{-1} = 5 \pmod 9$, $\epsilon(\eta) \in (F^+)^9$. (Note. In general if a lies in a proper subfield of F , then ϵa is at least an l th power in F but not necessarily an l' th power.) It follows that $\zeta = \epsilon a$ with $a \in F^*$ iff $\zeta = \pm \epsilon \zeta^i$ for some i ; this is impossible since $\epsilon \zeta \in F^3$, and the example is complete.

Return to the situation described above; $E/F^* = RH\alpha$. Let $\Sigma = \sum_i \sigma^i \in RH$ and let X_0 be the kernel of Σ acting on X . For any $x \in X$, $\langle \epsilon \alpha, x \rangle = \langle \alpha, x^2 \rangle$ so that $x \in \text{Gal}(N/F(\epsilon \alpha))$ iff $\langle \alpha, x^2 \rangle = 1$. Since x^2 is invariant under H , α generates E/F^* as an RH -module, and the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate, it follows that $\text{Gal}(N/F(\epsilon \alpha)) = X_0$. The preceding proposition expresses the fact that $X^{1-\sigma} \subseteq X_0$. Let $L = F(\epsilon \alpha)$; it remains to decide under what circumstances L contains a subfield K which is cyclic of degree l' over k . Of course $\text{Gal}(L/k)$ is an extension of H by a cyclic group of order a divisor of l' .

Proposition. *If $\alpha_0, \dots, \alpha_{m-1}$ are independent over R , then $\text{Gal}(L/k) \cong G \times H$.*

Proof. In this case E/F^* is the free RH -module with basis $\{\alpha\}$ and one deduces easily that $X \cong RH$ as well. (In fact X is generated as an RH -module by the automorphism which sends α to $\zeta \alpha$, the action of this automorphism on other elements of E being determined by the action of H on X and E/F^* .) It follows that $\mathfrak{G} \cong X \cdot H$, $X_0 = X^{1-\sigma}$, and $X/X_0 \cong G$, from which the assertion follows.

The hypothesis in the last result can be stated in terms of the a_i 's; namely that $a_0, \dots, a_{n-1}$ should be multiplicatively independent modulo l' th powers in F^* .

If this hypothesis is not satisfied it may happen that a field $L = F(\epsilon \alpha)$ is itself cyclic as an extension of k . For example, if $F = Q(\zeta)$, where ζ is a primitive l' th root of unity and $L = F((\epsilon \zeta)^{1/l'})$, this is the case.

From the last proposition one sees that the field L will contain subfields which are cyclic of degree l' over k ; the final task is to show that

when N/F is tamely ramified at least one of these extensions is tamely ramified over k .

Proposition. Assume $\alpha_0, \dots, \alpha_{n-1}$ are independent over R and assume N/F is tamely ramified. Let K be the subfield of N fixed by $X_0 \cdot H$, then K/k is a tamely ramified extension of type G .

Remark. The reader may wonder why the field L was ever mentioned since K is presented as a subfield of N . Of course $K \subseteq L$ and it will be useful in the next paragraph to dispense with N .

Proof. Suppose $V = \langle \sigma^h \rangle$ is the (first) ramification group of some prime ℓ dividing (l) for the extension F/k . The ramification group in N/k of any prime $\ell | \ell$ will be generated by an element of the form $x \cdot \sigma^h$ whose order, say ℓ^n , is the same as that of σ^h . In terms of the right action of H on X described above the multiplication in $X \cdot H$ is given by $(x_1 \cdot \sigma^i)(x_2 \cdot \sigma^j) = x_1 x_2^{\sigma^{-i}} \cdot \sigma^{i+j}$. Hence $1 + \sigma^{-h} + \sigma^{-2h} + \dots + \sigma^{-(\ell^n-1)h}$ annihilates x . Since X is free as a V -module it follows that $x \in X^{1-\sigma^h} \subseteq X^{1-\sigma} = X_0$, and the proposition is proven.

3. The results of §2 make it possible to prove the following

Theorem. $R_t(k, G) = W^{(l-1)/2}$. Given $A \in W$ and any ideal $\mathfrak{b}$ in O_k , there is a tamely ramified extension K/k of type G for which $C(K, k) = A$ and $(\mathfrak{b}_{K/k}, \mathfrak{b}) = 1$.

Proof (compare Theorem (2.6) in [4]). Let $t > 3$ be an odd integer such that $A^t = A$ and choose positive integers $h_1, \dots, h_t$ relatively prime to l so that $\sum h_i = lt$. Choose primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ relatively prime to $\mathfrak{b}$ in F each of which splits completely in F/k , no two of which are conjugate in F/k , and so that each $\mathfrak{p}_i \cap k$ is in the class A . Assume also that all the $\mathfrak{p}_i$'s are in the same ideal class modulo a sufficiently high power of l and let $q_i, i = 1, \dots, t$, be nonconjugate prime ideals in the inverse class. Then the ideal

$$\mathfrak{p}_1^{h_1} \dots \mathfrak{p}_t^{h_t} (q_1 \dots q_t)^l$$

is a principal ideal generated by some $a' \in F$ which is congruent to 1 modulo a high power of l . To insure that the conjugates of a' will be multiplicatively independent modulo l 'th powers in F let $a = \rho a'$, where (ρ) is a prime ideal not dividing $\mathfrak{b}$ which splits completely in F/k , and $\rho \not\equiv 1$ modulo a high power of l . The conditions satisfied by a insure that the con-

jugates $\sigma^i(a)$ are independent over R (acting exponentially) and that the field $N(a)$ described in §2 is tamely ramified over F .

(Indeed, if $\mathfrak{l}$ is a prime divisor of (l) in F , the condition $a \equiv 1 \pmod{a}$ high power of l means that a is an l' th power in the $\mathfrak{l}$ -adic completion of F , so that $\mathfrak{l}$ splits completely in N/F . The expression "a high power of l " means any power n so high that, for each $\mathfrak{l} | (l)$ in F , $a \equiv 1 \pmod{\mathfrak{l}^n}$ implies a is an l' th power in $F_{\mathfrak{l}}$.) Now let K be the subfield of $N(a)$ described in §2. Then $L = KF$ and $\mathfrak{b}_{L/F} = \mathfrak{b}_{K/k}$ as ideals in O_F . Let $P_i = \prod_j \sigma^j(\mathfrak{p}_i)$, $Q_i = \prod_j \sigma^j(\mathfrak{q}_i)$; these are prime ideals in O_k . It is easy to see that

$$\mathfrak{b}_{L/F} = \prod_i P_i^{l'-1} Q_i^{l(l'-1)} \cdot (\rho)^n$$

for some n . From this it follows that $C(K, k) = A^{(l-1)/2}$.

4. In general $R(k, G)$ will be a union of cosets of $R_{\mathfrak{l}}(k, G)$ within the ideal class group $C(k)$. For a given field k it can be described explicitly in terms of the structure of the groups of principal units in the completions $k_{\mathfrak{l}}$ of k at primes $\mathfrak{l} | (l)$. More specifically suppose that $\mathfrak{l}_1, \dots, \mathfrak{l}_g$ are the primes of k which divide (l) . Suppose that for each i there is given an ideal $\mathfrak{l}_i^{e_i}$ which is the discriminant of some extension of type G of the local field $k_i =$ completion of k at $\mathfrak{l}_i$. Then by Grunwald's theorem there is an extension K_1/k of type G with discriminant $\prod_i \mathfrak{l}_i^{e_i} \cdot \mathfrak{b}$ where $(\mathfrak{b}, (l)) = 1$. (This depends on the fact that $l > 2$.) From the proof of the proposition in §1 it follows that $C(\mathfrak{b}) \in W^{(l-1)/2}$. Now suppose $A \in W$. By the theorem there is a tamely ramified extension K_2/k of type G whose discriminant is relatively prime to $(l)\mathfrak{b}$ and for which

$$C(K_2, k) = (A \cdot C(\mathfrak{b})^{-1})^{(l-1)/2}.$$

The composite $K_1 K_2$ contains a subfield K which is of type G over k and for which $\mathfrak{b}_K = \mathfrak{b}_1 \mathfrak{b}_2$. ($\mathfrak{b}_i$ is the discriminant of K_i/k . One can take for K the field fixed by the "diagonal subgroup" of $\text{Gal}(K_1 K_2/k)$, cf. Lemma (4.3) in [4].) It follows that

$$C(K, k) = A^{(l-1)/2} \cdot \prod_i C(\mathfrak{l}_i)^{e_i/2}$$

so that $R(k, G)$ is determined by which exponents e_i may occur. An example of a field k and a group G , elementary abelian of type $(3, 3)$, for which $R(k, G)$ is not a group is given in [5]. Of course when G is cyclic of order l , $R(k, G)$ proves to be a group. I do not yet know exactly what happens in the case G cyclic of order l' .

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