AN EXTENSION OF THE DENJOY INTEGRAL

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ABSTRACT. The purpose of this paper is to exhibit an additive class of continuous functions such that each function of the class satisfies Lusin's condition (N). Restricting this class to those functions which are approximately derivable almost everywhere produces a class of primitives which is more extensive than the class of ACG functions, the primitives for the wide sense Denjoy integral.

Let $\mathcal{F}$ be any additive class of continuous functions such that each function $F \in \mathcal{F}$ satisfies Lusin's condition (N). Let $\mathcal{D}$ be an additive class of continuous functions derivable in a sense which is compatible with the ordinary derivative $F'(x)$; i.e., $DF(x) = F'(x)$ at almost every point where $F'(x)$ exists. For each pair of functions $F, G \in \mathcal{D}$, suppose that D satisfies $D(F - G) = DF - DG$ almost everywhere. Then $\mathcal{F} \cap \mathcal{D}$ can be taken as a class of primitives and the $\mathcal{F}\mathcal{D}$ -integral can be defined by

$$\mathcal{F}\mathcal{D} \int_a^b f(x)dx = F(b) - F(a)$$

where $DF(x) = f(x)$ almost everywhere on $[a, b]$. The uniqueness of the integration is implied by the following theorem:

If a continuous function F satisfies Lusin's condition (N) on an interval I and $F'(x)$ is nonnegative at almost every point x where $F'(x)$ exists, then F is monotone nondecreasing on I (cf. [1, pp. 285–286]).

Then if $DF = DG$ almost everywhere on I , it follows that $D(F - G) = DF - DG = 0 = D(G - F)$ almost everywhere on I and thus $F - G$ is a constant (the theorem implies that both $F - G$ and $G - F$ are monotone nondecreasing). Consequently

$$\mathcal{F}\mathcal{D} \int_a^b f(x)dx = F(b) - F(a) = G(b) - G(a)$$

and the integration yields a unique value.

In what follows $DF(x)$ will be the approximate derivative: $F'_{\text{ap}}(x_0) = L$

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if there is a set A_{x_0} so that the density of A_{x_0} at x_0 equals 1 and

$$\lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} = L$$

when the limit is taken with respect to points x in A_{x_0} . $\mathcal{D}$ will consist of the continuous functions which are approximately derivable almost everywhere. (Alternative differentiation processes could be considered here and hopefully one can be found that includes every function in $\mathcal{F}$.)

The following definitions are preliminary to defining the class $\mathcal{F}$.

Definition 1. Given a natural number N and a set E , a function f will be said to be $B(N)$ on E if there is a number $M < \infty$ such that for any sequence $I_1, \dots, I_k, \dots$ of nonoverlapping intervals with $I_k \cap E \neq \emptyset$, there exist intervals J_{kn} , $n = 1, 2, \dots, N$, such that

$$B\left(f; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^N |J_{kn}| < M.$$

(Here, $B(f; X)$ is the graph of F on the set X .)

Definition 2. Given a natural number N and a set E , a function f will be said to be $A(N)$ on E if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $I_1, \dots, I_k, \dots$ are nonoverlapping intervals with $E \cap I_k \neq \emptyset$ and $\sum_k |I_k| < \delta$, then there exist intervals J_{kn} , $n = 1, 2, \dots, N$, such that

$$B\left(f; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^N |J_{kn}| < \epsilon.$$

The class $\mathcal{F}$ will consist of all continuous functions F defined on closed interval I for which there exist a sequence of sets E_n and natural numbers N_n such that $I = \bigcup_n E_n$ and F is $A(N_n)$ on E_n .

From Definitions 1 and 2 it follows that:

(i) In the event that $N = 1$ or E is an interval, if F is a continuous function satisfying $A(N)$ [$B(N)$] on E , then f is absolutely continuous [of bounded variation] on E .

(ii) If a continuous function f satisfies $A(N)$ [$B(N)$] on E , then f satisfies $A(N)$ [$B(N)$] on $\bar{E}$ = the closure of E , and on any subset of $\bar{E}$.

(iii) If a function f satisfies $A(N)$ on E and Z is a set of Lebesgue measure 0 contained in E , then $\Lambda(B(f; Z)) = 0$ (where $\Lambda(x)$ is the linear measure of X). Consequently, the Lebesgue measure of $f(Z)$ is also 0.

(iv) If a function f is $B(N)$ on E with $|E| < \infty$ then $\Lambda(B(f; E)) < \infty$.

(v) If a function f is continuous and satisfies $A(N)$ on a bounded set E then f satisfies $B(N)$ on E .

(vi) If a function f_1 satisfies $A(N) [B(N)]$ on E and a function f_2 satisfies $A(N') [B(N')]$ on E then every linear combination of f_1 and f_2 satisfies $A(N \cdot N') [B(N \cdot N')]$ on E . If f_1 and f_2 are bounded on E then $f_1 \cdot f_2$ also satisfies $A(N \cdot N') [B(N \cdot N')]$ on E .

Proof of (i). If $N = 1$ then Definition 1 is equivalent to the definition of absolute continuity on a set E and Definition 2 is equivalent to the definition of bounded variation on a set E . If E is an interval, since f is continuous, the graph of f is connected and $\bigcup_{n=1}^N J_{kn}$ obtained by the definition of $A(N) [B(N)]$ is an interval (for each k). Consequently

$$\sum_k O(f; I_k) \leq \sum_k \left(\sum_{n=1}^N |J_{kn}| \right) < \epsilon$$

whenever $\sum |I_k| < \delta$, where $O(f; I)$ is the oscillation of f on I . Thus f is absolutely continuous on the interval E [$\sum_k O(f; I_k) \leq \sum_k (\sum_{n=1}^N |J_{kn}|) < M < \infty$ and f is of bounded variation on the interval E].

Proof of (ii). If f is continuous and satisfies $A(N) [B(N)]$ on E , it is sufficient that the definitions be satisfied for every finite sequence of closed intervals I_k . But then $B(f; E \cap \bigcup I_k) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn}$ holds true if E is replaced by $\bar{E}$. Thus f satisfies $A(N) [B(N)]$ on $\bar{E}$ and also on any subset of $\bar{E}$.

Proof of (iii). Let $Z \subset E$ be a set of measure 0 and f be $A(N)$ on E . For any $\epsilon > 0$ there is a $\delta > 0$ such that if $I_1, \dots, I_k, \dots$ is a sequence of nonoverlapping intervals with $I_k \cap E \neq \emptyset$ and $\sum |I_k| < \delta$ there exist J_{kn} , $n = 1, 2, \dots, N$, such that

$$B(f; E \cap \bigcup I_n) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^N |J_{kn}| < \epsilon.$$

Select a sequence of intervals $I_1, \dots, I_k, \dots$ such that $Z \subset \bigcup I_k$ and $\sum |I_k| < \min(\epsilon, \delta)$. It follows that

$$\begin{aligned} \sum_{k,n} \text{diam}(I_k \times J_{kn}) &\leq \sum_{k,n} |I_k| + |J_{kn}| = N \cdot \sum_k |I_k| + \sum_{k,n} |J_{kn}| \\ &< N\delta + \epsilon \leq (N+1)\epsilon. \end{aligned}$$

Since ϵ is arbitrary, it follows that $\Lambda(B(f; Z)) = 0$.

Proof of (iv). Given $\epsilon > 0$, let $I_1, \dots, I_k, \dots$ be a sequence of non-

overlapping intervals containing E such that $|I_k| < \epsilon$ and $|\Sigma I_k| - |E| < \epsilon$. Then there are intervals J_{kn} , $n = 1, 2, \dots, N$, such that

$$B(f; E) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^N |J_{kn}| < M < \infty,$$

where M is the bound dependent on f . Each interval J_{kn} can be written as the union of at most $[|J_{kn}|/|I_k|] + 1$ intervals J_{knm} of length $|I_k|$ (where $[x]$ is the greatest integer less than x). But then

$$\begin{aligned} \sum \text{diam}(I_k \times J_{knm}) &\leq \sum_{n,k,m} (|I_k| + |J_{knm}|) \leq \sum_{k,n} 2 \left(\frac{|J_{kn}|}{|I_k|} + 1 \right) |I_k| \\ &\leq \sum_{k,n} 2(|J_{kn}| + |I_k|) \leq 2(M + N(|E| + \epsilon)). \end{aligned}$$

Since ϵ is arbitrary, $\Lambda(B(f; E)) \leq 2M + 2N|E| < \infty$.

Proof of (v). From (ii) and the hypotheses of (v), f satisfies $A(N)$ on the compact set $\bar{E}$. For $\epsilon = 1$ let $\delta > 0$ be the δ guaranteed by the definition of $A(N)$. For each $x \in E$ let $U(x)$ be the open interval $(x - \delta/2, x + \delta/2)$. Since the $U(x)$ form an open cover of $\bar{E}$ there is a finite subcover $U_1(x), \dots, U_M(x)$. If $|f(x)| < K$ on $\bar{E}$ then, whenever $I_1, \dots, I_k, \dots$ are pairwise nonoverlapping intervals with $I_k \cap E \neq \emptyset$, let $\{I_m\}$ be those intervals contained in $U_m(x)$. There are J_{mn} , $n = 1, 2, \dots, N$, such that

$$B\left(f; E \cap \bigcup_m I_m\right) \subset \bigcup_m \bigcup_{n=1}^N I_m \times J_{mn} \quad \text{and} \quad \sum_m \sum_{n=1}^N |J_{mn}| < 1.$$

For the intervals I_k which are contained in no $U_m(x)$ let $J_k^n = [-K, K]$ and note that there are at most $2M$ such intervals I_k . Then

$$B\left(f; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n=1}^N J_{kn} \times I_k \quad \text{and} \quad \sum_k \sum_{n=1}^N |J_{kn}| \leq 2K \cdot 2M + M < \infty.$$

Proof of (vi). If f_1 and f_2 are two functions and

$$B(f_1; E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn}$$

and

$$B(f_2; E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n'=1}^{N'} I_k \times J_{kn'}$$

then if

$$J_{knn'} = [a_{kn} + a_{kn'}, b_{kn} + b_{kn'}],$$

where $J_{kn} = [a_{kn}, b_{kn}]$ and $J_{kn'} = [a_{kn'}, b_{kn'}]$, it follows that

$$B(f_1 + f_2; E \cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^N \bigcup_{n'=1}^{N'} I_k \times J_{knn'}.$$

Since $|J_{knn'}| = |J_{kn}| + |J_{kn'}|$ it follows that

$$\sum_{k,n,n'} |J_{knn'}| = N' \sum_{k,n} |J_{kn}| + N \sum_{k,n} |J_{kn'}|.$$

Thus if f_1 and f_2 are $B(N)$ and $B(N')$ respectively on E then $f_1 + f_2$ is $B(N \cdot N')$ on E . In the case where f_1 is $A(N)$ and f_2 is $A(N')$ on E , given $\epsilon > 0$, let $\epsilon_1 = \epsilon/2N'$ and $\epsilon_2 = \epsilon/2N$. Let δ_1 be the δ determined by ϵ_1 and the fact that f_1 satisfies $A(N)$ on E ; let δ_2 be the δ determined by ϵ_2 and the fact that f_2 satisfies $A(N')$ on E ; let $\delta_0 = \min(\delta_1, \delta_2)$. Then, if $I_1, \dots, I_k, \dots$ are pairwise nonoverlapping intervals with $I_k \cap E \neq \emptyset$ and $\sum_k |I_k| < \delta_0$, there are J_{kn} , $n = 1, 2, \dots, N$, and $J_{kn'}$, $n' = 1, 2, \dots, N'$, such that

$$B\left(f_1; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n=1}^N I_k \times J_{kn} \quad \text{and} \quad \sum_{k,n} |J_{kn}| < \epsilon_1,$$

$$B\left(f_2; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n'=1}^{N'} I_k \times J_{kn'} \quad \text{and} \quad \sum_{k,n'} |J_{kn'}| < \epsilon_2.$$

Consequently

$$B\left(f_1 + f_2; E \cap \bigcup_k I_k\right) \subset \bigcup_k \bigcup_{n=1}^N \bigcup_{n'=1}^{N'} J_{knn'}$$

and

$$\sum_{k,n,n'} |J_{knn'}| < N'\epsilon_1 + N\epsilon_2 < \epsilon.$$

Thus $f_1 + f_2$ is $A(N \cdot N')$ on E . Since it is apparent that $k_1 f_1$ belongs to $A(N)$ and $k_2 f_2$ belongs to $A(N')$ (respectively $B(N)$ and $B(N')$) it follows that linear combinations of f_1 and f_2 belong to $A(N \cdot N')$ [$B(N \cdot N')$]. In the event that f_1 and f_2 are bounded, it is sufficient to consider that $0 \leq f_i(x) \leq A < \infty$. Then if

$$J_{knn'} = [a_{kn} \cdot a_{kn'}, b_{kn} \cdot b_{kn'}]$$

when $J_{kn} = [a_{kn}, b_{kn}]$ and $J_{kn'} = [a_{kn'}, b_{kn'}]$ it follows that

$$\begin{aligned} |J_{knn'}| &= b_{kn} \cdot b_{kn'} - a_{kn} \cdot a_{kn'} \\ &= b_{kn}(b_{kn'} - a_{kn'}) + a_{kn'}(b_{kn} - a_{kn}) \leq A|J_{kn}| + A|J_{kn'}|. \end{aligned}$$

The proof that $f_1 \cdot f_2$ is in $A(N \cdot N') [B(N \cdot N')]$ on E proceeds similarly to the proof for the sum of f_1 and f_2 .

One may note that each function F belonging to $\mathcal{F}$ on an interval I is absolutely continuous on some subinterval of I . (This follows from the definition of $\mathcal{F}$, (i), (ii), and the Baire category theorem.)

If F and G belong to $\mathcal{F}$ on I , there exist sets $\{E_n\}_{n=1}^\infty$, $\{E'_k\}_{k=1}^\infty$ and natural numbers $\{N_n\}_{n=1}^\infty$, $\{N'_k\}_{k=1}^\infty$ so that F is $A(N_n)$ on E_n , G is $A(N'_k)$ on E'_k and $I = \bigcup_n E_n = \bigcup_k E'_k$. From (vi) it follows that $F + G$ is $A(N_n \cdot N'_k)$ on $E_n \cap E'_k$. Since $I = \bigcup_{n,k} (E_n \cap E'_k)$ it follows that $F + G$ belong to $\mathcal{F}$ on I . Since (iii) implies that each function F belonging to $\mathcal{F}$ on I also satisfies *Lusin's condition (N)* on I , it follows that $\mathcal{F}$ is an additive class of functions each of which satisfies *Lusin's condition (N)*. From the preliminary remarks one concludes that $\mathcal{F} \cap \mathcal{D}$ can be taken as a class of primitives on an interval I .

In conclusion a member of $\mathcal{F}$ which is not ACG is constructed.

Example. Let $I = [0, 1]$ and let C denote the Cantor ternary set; i.e., $C = \{x | x = \sum c_i/3^i \text{ with } c_i = 0 \text{ or } c_i = 2 \text{ for each } i\}$. Each point $x \in C$ is uniquely represented by $\sum c_i(x)/3^i$. Let $\{j_k\}$ be a strictly increasing sequence of natural numbers and set $j_0 = 0$. Let $h(i) = j_{k-1}$ when $j_{k-1} \leq i < j_k$, and define F on C by $F(\sum c_i/3^i) = \sum c_{h(i)}/3^i$. Then F is continuous on C and, by extending F linearly on each interval contiguous to C , one has F defined and continuous on $[0, 1]$. Consider an interval I such that $I \cap C \neq \emptyset$ and $3^{-m-1} \leq |I| < 3^{-m}$ for a natural number m . Since $|I| < 3^{-m}$, there exist $c_1, c_2, \dots, c_m$ such that if $x \in I \cap C$ then $c_i(x) = c_i$, $i = 1, 2, \dots, m$. Let $A = \sum_{i=1}^m c_i/3^i$ and suppose that $h(m) = j_{k-1}$. Let J_1 and J_2 be two intervals defined by

$$\begin{aligned} J_1 &= \left[F(A), F\left(A + \sum_{i > j_k} 2/3^i\right) \right], \\ J_2 &= \left[F(A + 2/3^{j_k}), F\left(A + \sum_{i \geq j_k} 2/3^i\right) \right]. \end{aligned}$$

Then $F(I \cap C) \subset J_1 \cup J_2$ and $|J_1| = |J_2| = 3^{-jk} \leq 3^{-m} < 3|I|$. It now follows that F is $A(2)$ on C . For if $\{I_n\}$ is a sequence of nonoverlapping intervals with $I_n \cap C \neq \emptyset$ and $\sum |I_n| < \epsilon/6$, then there exist intervals J_{1n}, J_{2n} such that

$$B\left(F; C \cap \bigcup_n I_n\right) \subset \bigcup_n \bigcup_{k=1}^2 I_n \times J_{kn}$$

and

$$\sum_n \sum_{k=1}^2 |J_{kn}| \leq \sum_n \sum_{k=1}^2 3|I_n| \leq 6 \sum_n |I_n| < \epsilon.$$

Since F is $A(1)$ (absolutely continuous) on each interval contiguous to C , it follows that $F \in \mathcal{F}$. Appropriate choice of the numbers j_k results in an F which is not ACG. In particular, if j_{k+1} is chosen so large that $n_k = j_{k+1} - j_k - 1$ satisfies $2^{n_k} - 1 > k \cdot 3^{j_k}$, then F is not ACG. In order to show this, it suffices to show that F is of unbounded variation on every portion of C (cf. [1, p. 233]). To see this, let c_i be 0 or 2 for $i = 1, 2, \dots, j_k$. Let $a = \sum_{i=1}^{j_k} c_i$ and $b = a + \sum_{i=j_k+1}^{\infty} 2/3^i$. Then there is a sequence of 2^{n_k} numbers x in $I = [a, b]$ at which $F(x) = F(a)$ or $F(x) = F(b)$ in alternating fashion; namely, all the numbers x satisfying $c_i(x) = c_i$, $i = 1, 2, \dots, j_k$, and either $c_i(x) = 0$ for each $i \geq j_{k+1}$ or $c_i(x) = 2$ for each $i \geq j_{k+1}$. Since $F(b) - F(a) = 3^{-j_k}$ it follows that the variation of F on $I \cap C$ is greater than $(2^{n_k} - 1)3^{-j_k} > k$. Because any portion of C contains such an interval I , it follows that F is of unbounded variation on each portion of C and hence is not ACG.

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