SCHUR'S THEOREM FOR NEARLY KÄHLER MANIFOLDS

A. M. NAVEIRA AND L. M. HERVELLA

ABSTRACT. The classical theorem of Schur on Kähler manifolds is generalized to nearly Kähler manifolds, thus solving a conjecture of A. Gray [3, p. 289].

Let M be a C^∞ manifold with metric tensor g , Riemannian connection ∇ , and almost complex structure J . Denote by $F(M)$ the real valued C^∞ functions on M , by $\chi(M)$ the C^∞ vector fields of M , and by M_m the tangent space of M at m . Then M is said to be a nearly Kähler manifold provided $(\nabla_X J)X = 0 \quad \forall X \in \chi(M)$. In this paper we consider M as a nearly Kähler manifold.

Lemma 1 [2]. For all $W, X, Y, Z \in \chi(M)$, we have

$$R(X, Y, X, Y) - R(X, Y, JX, JY) = \|(\nabla_X J)Y\|^2,$$

$$R(W, X, Y, Z) = R(JW, JX, JY, JZ).$$

Lemma 2 [3]. For all $W, X, Y, Z \in \chi(M)$, we have

$$R(W, X, Y, Z) - R(W, X, JY, JZ) = g\{(\nabla_W J)X, (\nabla_Y J)Z\}.$$

Proposition 3. Let R and T be two quadrilinear mappings satisfying

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = -R(X, Y, W, Z) = R(Z, W, X, Y),$$

$$(I) \quad R(X, Y, Z, W) + R(X, W, Y, Z) + R(X, Z, W, Y) = 0,$$

$$R(X, Y, Z, W) - R(JX, JY, Z, W) = g\{(\nabla_X J)Y, (\nabla_Z J)W\}.$$

If $R(X, JX, X, JX) = T(X, JX, X, JX)$ for all $X \in M_m$, then

$$\begin{aligned} T(X, Y, Z, W) = & R(X, Y, Z, W) \\ & \pm [\frac{1}{2}g\{(\nabla_X J)Y, (\nabla_Z J)W\} + \frac{1}{4}g\{(\nabla_Z J)Y, (\nabla_X J)W\} \\ & - \frac{1}{4}g\{(\nabla_X J)Z, (\nabla_W J)Y\}]. \end{aligned}$$

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Proof. We may assume that $T = 0$. Consider $R - T$ and 0 instead of R and T . Consider the quadrilinear mapping which sends $(X, Y, Z, W) \in M_m$ into

$$R(X, JY, Z, JW) + R(X, JZ, Y, JW) + R(X, JW, Y, JZ) \\ - \frac{1}{2}g\{(\nabla_X J)Y, (\nabla_Z J)W\} - \frac{1}{2}g\{(\nabla_X J)Z, (\nabla_Y J)W\} - \frac{1}{2}g\{(\nabla_X J)W, (\nabla_Y J)Z\}.$$

This quadrilinear mapping is symmetric in X, Y, Z and W . Since it vanishes for $X = Y = Z = W$ by the assumption that $T = 0$, it must vanish identically. Setting $X = Z$ and $Y = W$, we obtain

$$2R(X, JY, X, JY) + R(X, JX, Y, JY) = g\{(\nabla_X J)Y, (\nabla_X J)Y\}.$$

From (I), we have

$$(II) \quad 3R(X, JY, X, JY) + R(X, Y, X, Y) = 3g\{(\nabla_X J)Y, (\nabla_X J)Y\}.$$

Replacing Y by JY in (II) and operating, we obtain

$$(III) \quad -9R(X, Y, X, Y) - 3R(X, JY, X, JY) = -9g\{(\nabla_X J)Y, (\nabla_X J)Y\}.$$

From (II) and (III), we obtain

$$R(X, Y, X, Y) = \frac{3}{4}g\{(\nabla_X J)Y, (\nabla_X J)Y\}.$$

By a typical method we have

$$(IV) \quad R(X, Y, Z, W) = \frac{1}{2}g\{(\nabla_X J)Y, (\nabla_Z J)W\} \\ + \frac{1}{4}g\{(\nabla_Z J)Y, (\nabla_X J)W\} - \frac{1}{4}g\{(\nabla_X J)Z, (\nabla_W J)Y\}.$$

Now we define the quadrilinear mapping R'_0 as follows:

$$(V) \quad R'_0(X, Y, Z, W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(X, JZ)g(Y, JW) \\ - g(X, JW)g(Y, JZ) + 2g(X, JY)g(Z, JW) \\ + \frac{1}{2}g\{(\nabla_X J)Y, (\nabla_W J)Z\} - \frac{1}{4}g\{(\nabla_Z J)Y, (\nabla_X J)W\} \\ + \frac{1}{4}g\{(\nabla_X J)Z, (\nabla_W J)Y\}.$$

Proposition 4. *The quadrilinear mapping R'_0 satisfies (I), and*

$$R'_0(X, JX, X, JX) = \|X\|^2.$$

Proposition 5. *Let P be a plane at M ; let X, Y be an orthonormal*

basis for P ; $K(P)$ their sectional curvature; R a quadrilinear mapping satisfying (I). If $K(P)$ is constant for all planes P , invariant by J , then

$$R(X, Y, Z, W) = KR'_0(X, Y, Z, W) \\ \pm [\frac{1}{2}g\{(\nabla_X J)Y, (\nabla_Z J)W\} + \frac{1}{4}g\{(\nabla_Z J)Y, (\nabla_X J)W\} \\ - \frac{1}{4}g\{(\nabla_X J)Z, (\nabla_W J)Y\}].$$

The proof is immediate from Lemma 2 and Proposition 3.

Theorem 6. Let M be a connected nearly Kähler manifold of complex dimension $n \geq 2$. If the holomorphic sectional curvature $K(P)$, where P is a plane in M_m invariant by J , depends only on m , then M is a space of constant holomorphic sectional curvature.

Proof. We denote by $\alpha(X, Y, Z, W)$ the tensor

$$\alpha(X, Y, Z, W) = \frac{1}{2}g\{(\nabla_X J)Y, (\nabla_Z J)W\} \\ + \frac{1}{4}g\{(\nabla_Z J)Y, (\nabla_X J)W\} - \frac{1}{4}g\{(\nabla_X J)Z, (\nabla_W J)Y\}.$$

By Proposition 5, we know that

$$(VI) \quad R(X, Y, Z, W) = KR'_0(X, Y, Z, W) \pm \alpha(X, Y, Z, W) \\ = KR_0(X, Y, Z, W) - (K \pm 1)\alpha(X, Y, Z, W),$$

where

$$R_0(X, Y, Z, W) = g(X, Z)g(Y, W) - g(X, W)g(Y, Z) + g(X, JW)g(Y, JZ) \\ - g(X, JZ)g(Y, JW) + 2g(X, JY)g(Z, JW).$$

From (VI), we obtain

$$(VII) \quad (\nabla_U R)(X, Y, Z, W) = (UK)R_0(X, Y, Z, W) + K(\nabla_U R_0)(X, Y, Z, W) \\ - (UK)\alpha(X, Y, Z, W) - (K \pm 1)(\nabla_U \alpha)(X, Y, Z, W).$$

If we make the cyclic sum in (VII) with respect to (U, X, Y) and use Bianchi's second identity, we obtain

$$0 = (UK)(R_0 - \alpha)(X, Y, Z, W) \\ + (YK)(R_0 - \alpha)(U, X, Z, W) + (XK)(R_0 - \alpha)(Y, U, Z, W) \\ + K\{(\nabla_U R_0)(X, Y, Z, W) + (\nabla_Y R_0)(U, X, Z, W) + (\nabla_X R_0)(Y, U, Z, W)\} \\ - (K \pm 1)\{(\nabla_U \alpha)(X, Y, Z, W) + (\nabla_Y \alpha)(U, X, Z, W) + (\nabla_X \alpha)(Y, U, Z, W)\}.$$

For an arbitrary Y , we choose X such that X, Y, JY are mutually orthogonal to Y, JY and X . Besides we can choose U, Z and W such that $U = Z = JX, W = X$. Then

$$\begin{aligned} 0 &= 4(YK)g(X, X)^2 \\ &+ K\{(\nabla_{JX}R_0)(X, Y, JX, X) + (\nabla_YR_0)(JX, X, JX, X) \\ &\quad + (\nabla_XR_0)(Y, JX, JX, X)\} \\ &- (K \pm 1)\{(\nabla_{JX}\alpha)(X, Y, JX, X) + (\nabla_Y\alpha)(JX, X, JX, X) \\ &\quad + (\nabla_X\alpha)(Y, JX, JX, X)\}. \end{aligned}$$

By using $(\nabla_XJ)X = 0$ and $g((\nabla_YJ)X, X) = 0$ for Y arbitrary, from

$$\begin{aligned} (\nabla_UR_0)(X, Y, Z, W) &= \nabla_UR_0(X, Y, Z, W) - R_0(\nabla_UX, Y, Z, W) \\ &- R_0(X, \nabla_UY, Z, W) - R_0(X, Y, \nabla_UZ, W) \\ &- R_0(X, Y, Z, \nabla_UW) \end{aligned}$$

and

$$\begin{aligned} (\nabla_U\alpha)(X, Y, Z, W) &= \nabla_U\alpha(X, Y, Z, W) - \alpha(\nabla_UX, Y, Z, W) \\ &- \alpha(X, \nabla_UY, Z, W) - \alpha(X, Y, \nabla_UZ, W) \\ &- \alpha(X, Y, Z, \nabla_UW), \end{aligned}$$

by substituting and operating, we prove that

$$\begin{aligned} (\nabla_{JX}R_0)(X, Y, JX, X) &= (\nabla_YR_0)(JX, X, JX, X) \\ &= (\nabla_XR_0)(Y, JX, JX, X) = 0 \end{aligned}$$

and

$$(\nabla_{JX}\alpha)(X, Y, JX, X) = (\nabla_Y\alpha)(JX, X, JX, X) = (\nabla_X\alpha)(Y, JX, JX, X) = 0.$$

Then $(YK) = 0$.

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DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA, UNIVERSIDAD DE SANTIAGO,
SANTIAGO DE DOMPOSTELA, SPAIN