ON A GAME THEORETIC NOTION OF COMPLEXITY FOR COMPACT CONVEX SETS

EHUD KALAI AND MEIR SMORODINSKY

ABSTRACT. The notion of complexity for compact convex sets introduced by Billera and Bixby is considered. It is shown that for $n \geq 3$ there are sets in R^n of complexity n . Also for $n = 3$ the maximal complexity is 3.

1. Introduction. The following definitions occur in the mathematical model of market games. Let $u_1, u_2, \dots, u_n$ be n concave continuous real valued functions defined on the m -dimensional Euclidean unit cube I^m .

An *allocation* (of the commodities) is a matrix $(x) = (x_i^j)$, $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$, satisfying the conditions:

- (1) $x_i^j \geq 0$ for every i and j , and
- (2) $\sum_{j=1}^n x_i^j = 1$ for $i = 1, 2, \dots, m$.

We define the function u from the set of allocations to R^n by

$$u((x_i^j)) = (u_1(x_1^1, x_2^1, \dots, x_m^1), \dots, u_n(x_1^n, x_2^n, \dots, x_m^n)).$$

The image of u is a compact subset of R^n (see [1]). A subset $V \subset R^n$ is *attainable* if for some u , $V = \{p \in R^n: p \leq u(x) \text{ for some allocation } (x)\}$.

(For $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$, we say that $p \leq q$ if $p_i \leq q_i$ for $i = 1, 2, \dots, n$.) Another way to define V is $V = \text{Image}(u) - R_+^n$. ($R_+^n = \{x = (x_1, \dots, x_n) | x_i \geq 0, i = 1, \dots, n\}$.)

Intuitively we think of n traders ("players") in a market of m commodities. $u_i(x_1, x_2, \dots, x_m)$ represents the *utility* that the j th player has if he holds the *bundle of commodities* $(x_1, x_2, \dots, x_m)$, where x_i represents his portion of the total amount available of the i th commodity.

In a recent paper by Billera and Bixby it was shown that for any compact convex set $C \subset R^n$, the set $V = C - R_+^n$ is attainable for some $(u_1, u_2, \dots, u_n)$ on I^m , where m can be chosen so that $m \leq n(n-1)$. They defined the *com-*

Received by the editors November 13, 1973 and, in revised form, May 13, 1974 and May 24, 1974.

AMS (MOS) subject classifications (1970). Primary 90A15, 52A20; Secondary 90D12.

Key words and phrases. Compact convex sets, concave utility functions, complexity, market games.

Copyright © 1975, American Mathematical Society

plexity of V (or C) to be the minimal m for which this is true, and the complexity of n ($\text{com}(n)$) to be the maximal complexity of an attainable set in R^n .

It follows from the work of Billera and Bixby that $\text{com}(n) \leq n(n-1)$. It is easy to see that $\text{com}(2) = 1$, and in their paper they gave an example of an attainable set in R^3 with complexity 2, i.e. $2 \leq \text{com}(3) \leq 6$. One of them conjectured that $\text{com}(n) = n-1$ for all n .

In §2 we prove that for every n , $n \geq 3$, there is an attainable set of complexity n and, therefore, $\text{com}(n) \geq n$ for $n \geq 3$.

In §3 we show that $\text{com}(3) = 3$.

We are grateful to Nimrod Megiddo who made valuable comments about the problem.

2. Complexity in R^n .

Lemma 1. Suppose V_n^* is an attainable set in R^n , $n \geq 3$, satisfying the following properties:

(a) the $n+1$ points

$$\begin{aligned} p_0^* &= (\alpha, \alpha, \dots, \alpha), & p_1^* &= (1, 0, \dots, 0), \\ p_2^* &= (0, 1, 0, \dots, 0), \dots, & p_n^* &= (0, \dots, 0, 1) \end{aligned}$$

belong to V_n^* with $0 < \alpha < 1$;

(b) for each $p = (y_1, \dots, y_n) \in V_n^*$, if for some j , $1 \leq j \leq n$, $y_j > \alpha$, then $y_k < \alpha$ for every $k \neq j$.

Then $\text{com } V_n^* \geq n$.

Proof. Assume that $u = (u_1, \dots, u_n)$ are the utility functions on I^m that generate V_n^* . Let $(x) = (x_i^j)$ be the allocation for which $u(x) = p_0^*$, i.e. $u_j(x_1^j, \dots, x_m^j) = \alpha$ for $j = 1, 2, \dots, n$. Consider the bundle of the n th player in this allocation, $(x_1^n, \dots, x_m^n)$. Let J be the set of indices i for which $x_i^n = 1$. If $J \neq \emptyset$, assume without loss of generality that $J = \{k+1, \dots, m\}$.

We claim that there is an index i_1 , $1 \leq i_1 \leq k$, such that $x_{i_1}^j = 0$ for every j , $j \neq 1, n$. If not, put $\delta_i = \sum_{j \neq 1, n} x_i^j > 0$ for $1 \leq i \leq k$ and $\delta = \min_{1 \leq i \leq n} \delta_i$. Consider, for the n th player, the set of bundles

$$B_n = \{(x_1^n + r_1, x_2^n + r_2, \dots, x_m^n + r_m) : |r_l| < \delta \text{ for } l = 1, 2, \dots, m\} \cap I^m;$$

B_n is an open set in the relative topology of I^m . Since all the bundles of B_n can be allocated to the n th player without changing the bundle of the

first player, it follows from property (b) that u_n is bounded by α on B_n . So u_n has a local maximum in B_n and by concavity it must be a global maximum on I^n . But this contradicts the requirement that p_n^* is attained and establishes the truth of the claim.

Observe that $x_{i_1}^1 > 0$, for if $x_{i_1}^1 = 0$ then $x_{i_1}^n = 1$ and $i_1 > k$, which is not the case. Next, repeat the above argument with the 2nd and n th players instead of the 1st and n th players to conclude that there is an index i_2 , $1 \leq i_2 \leq k$, such that $x_{i_2}^j = 0$ for every j , $j \neq 2, n$, and $x_{i_2}^2 > 0$. Notice this implies $i_1 \neq i_2$. Continuing inductively, we find $i_1, \dots, i_{n-1}$ all distinct and all between 1 and k such that $x_{i_l}^j = 0$ for every $j \neq l, n$. This implies immediately that $m \geq k \geq n - 1$. Observe that for each player $j \neq n$, there are $n - 2$ commodities from which he gets zero quantity.

Suppose that $m = n - 1$ so that $J = \emptyset$. If this is the case, assign the roll of player n to player 1 to conclude that there is an index i_n such that $x_{i_n}^n > 0$ and $x_{i_n}^j = 0$ for every j , $j \neq 1, n$. This implies that $i_n = i_1$. But since the choice of player 1 is arbitrary, we can also conclude that $i_n = i_2$. Since $n \geq 3$, this is a contradiction, and the proof of the Lemma is completed.

Remark. It follows from the proof of Lemma 1 that in the case where $m = n > 3$, the allocation for the point p_0^* is the unit matrix up to a renaming of the commodities.

Theorem 1. *The attainable set $\bar{V}_n$ in R^n , $n \geq 3$, generated by the convex hull of the $n + 1$ points*

$$\begin{aligned} p_0^* &= (\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}), & p_1^* &= (1, 0, 0, \dots, 0), \\ p_2^* &= (0, 1, 0, \dots, 0), \dots, & p_n^* &= (0, \dots, 0, 1) \end{aligned}$$

has complexity n .

Proof. To show that the complexity of $\bar{V}_n$ is at least n , it is enough to show that $\bar{V}_n$ satisfies condition (b) of Lemma 1. Let p be a point in the convex hull of $\{p_0^*, \dots, p_n^*\}$; then $p = \sum_{l=0}^n \lambda_l p_l^*$, where $\lambda_l \geq 0$ and $\sum_{l=0}^n \lambda_l = 1$. Denoting $p = (y_1, \dots, y_n)$, we then have for $i \neq j$,

$$y_i + y_j = \lambda_0 + \lambda_i + \lambda_j \leq 1,$$

which proves property (b).

To show that $\text{com } \bar{V}_n \leq n$, we exhibit n utility functions from I^n to R that generate $\bar{V}_n$. Define

$$u_i(x_1, \dots, x_n) = \frac{1}{2} \left[x_i + \min_{1 \leq j \leq n} x_j \right] \quad \text{for } i = 1, \dots, n.$$

Denote by $\tilde{V}_n$ the attainable set of $u = (u_1, \dots, u_n)$. p_0^* is attained at the allocation that assigns to each player j the total amount of the j th commodity. For every player j , p_j^* is attained at the allocation that assigns to player j the total amount of all the commodities. Therefore $\tilde{V}_n \supset \bar{V}_n$. To show the converse inclusion, we start with an arbitrary allocation $x = (x_i^j)$ and show that the image of x under u is a point $p = (y_1, \dots, y_n)$ which is less than a point in the convex hull of $\{p_0^*, \dots, p_n^*\}$. Without loss of generality assume that for every player j , $x_i^j = \beta_j$ if $i \neq j$, $x_j^j = \alpha_j$ and $\alpha_j \geq \beta_j$.

Consider the point p' which is the image of the allocation that assigns to every player different from 1 and 2 the same amounts as in x , assigns to player 1 $(\alpha_1 - \beta_1, 0, \dots, 0)$ and assigns to player 2 $(\beta_2 + \beta_1, \alpha_2 + \beta_1, \beta_2 + \beta_1, \dots, \beta_2 + \beta_1)$. Also consider the point p'' obtained in the same way as p' , by interchanging the roles of players 1 and 2. p is a convex linear combination of p' and p'' . Now starting from p' and p'' and making the changes in the bundles of the 2nd player (or 1st in the case of p'') and the 3rd player, and so on, we finally arrive at allocations where all the off diagonal entries are zero except for one row. We infer that p is a convex linear combination of the images of such allocations of the following form: $x_j^j = \gamma$ and $x_i^j = 0$ if $i \neq j$ for every $j \neq k$; $x_k^k = 1$ and $x_i^k = 1 - \gamma$ for $i \neq k$ (where k is the row with the off diagonal nonzero entries). If $\tilde{p}$ is the image of such an allocation, then $\tilde{p} = (\tilde{y}_1, \dots, \tilde{y}_n)$, where $\tilde{y}_j = \frac{1}{2}\gamma$ if $j \neq k$ and $\tilde{y}_k = \frac{1}{2} + (\frac{1}{2} - \frac{1}{2}\gamma) = 1 - \frac{1}{2}\gamma$. So $\tilde{p} = \gamma p_0^* + (1 - \gamma)p_k^*$, and we conclude that p is a convex linear combination of $(p_0^*, \dots, p_n^*)$.

3. Complexity in R^3 . In this section we prove that any attainable set V in R^3 can be attained by three continuous, concave, monotone in each variable, utility functions on I^3 .

Theorem 2. $\text{com}(3) = 3$.

Proof. In view of Theorem 1 we have to show only that $\text{com}(3) \leq 3$.

Let C be a compact convex set in R^3 . Assume without loss of generality (see [1, p. 262]) that $\max\{y_1: (y_1, y_2, y_3) \in C \text{ for some } y_2, y_3\} = 1$ and that $\min\{y_1: (y_1, y_2, y_3) \in C \text{ for some } y_2, y_3\} = 0$. Also assume the same conditions for the 2nd and 3rd players. Let $V = C - R_+^3$. Consider the auxiliary function on I^2 ,

$$g(x_1, x_2) = \max\{\delta \in R: (x_1 + \delta, 1 - x_1 + \delta, 1 - x_2) \in V\}.$$

It is easy to verify that g is continuous and concave.

Now we shall define utility functions on I^3 and prove that these utility functions generate V .

$$u_1(x_1, x_2, x_3) = x_1 + g(x_1, x_2),$$

$$u_2(x_1, x_2, x_3) = x_1 + g(1 - x_1, x_3),$$

$$u_3(x_1, x_2, x_3) = \min\{x_2, x_3\}.$$

Let $\tilde{V}$ be the attainable set of $u = (u_1, u_2, u_3)$. To see that $\tilde{V} \supset V$, consider any point $y = (y_1, y_2, y_3) \in C$. Choose δ_0 such that $y_1 + \delta_0 + y_2 + \delta_0 = 1$. It follows that $0 \leq y_i + \delta_0 \leq 1$, $i = 1, 2$. Consider the allocation

$$x_1^1 = y_1 + \delta_0, \quad x_2^1 = 1 - y_3, \quad x_3^1 = 0,$$

$$x_1^2 = y_2 + \delta_0, \quad x_2^2 = 0, \quad x_3^2 = 1 - y_3,$$

$$x_1^3 = 0, \quad x_2^3 = y_3, \quad x_3^3 = y_3.$$

A straightforward calculation of $u((x_i^j))$ shows that $u((x_i^j)) = y$.

To prove that $\tilde{V} \subset V$, let $(x_i^j) = x$ be any allocation. It is easy to see that u_1, u_2, u_3 are monotonically increasing in each variable, so without loss of generality it can be assumed that $x_3^1 = 0$, $x_2^2 = 0$, $x_1^3 = 0$, and $x_2^3 = x_3^3$. If $\delta = g(x_1^1, 1 - x_3^3)$, then there is a point $(y_1, y_2, y_3) \in V$ such that $y_1 = x_1^1 + \delta$, $y_2 = 1 - x_1^1 + \delta$ and $y_3 = x_3^3$. But these y_i 's are the values of the u_i 's for the given allocation x .

REFERENCE

1. Louis J. Billera and Robert E. Bixby, *A characterization of Pareto surfaces*, Proc. Amer. Math. Soc. **41** (1973), 261-267.

DEPARTMENT OF STATISTICS, TEL AVIV UNIVERSITY, TEL AVIV, ISRAEL