THE BOUNDARY OF THE NUMERICAL RANGE

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ABSTRACT. This work includes two results which indicate a set theoretic relationship between the boundary of the numerical range and the essential numerical range. Several applications are derived.

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space with inner product $(\cdot, \cdot)$, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. The numerical range of $T \in \mathcal{B}(\mathcal{H})$ is by definition the set

$$W(T) = \{(Tx, x) : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well known that $W(T)$ is a bounded (not necessarily closed) convex subset of the plane. Furthermore, the closure of $W(T)$, denoted $W(T)^-$, contains the spectrum $\sigma(T)$ of T (see [4]).

The essential numerical range of T can be defined as the set

$$W_e(T) = \bigcap \{W(T + K)^- : K \in \mathcal{K}\},$$

where $\mathcal{K}$ denotes the ideal of compact operators on $\mathcal{H}$. Clearly $W_e(T)$ is a compact convex subset of the plane and $W_e(T) \subset W(T)^-$. It is known that $W_e(T)$ contains the essential spectrum $\sigma_e(T)$, which can be viewed as the spectrum of T represented canonically in the quotient $\mathcal{B}(\mathcal{H})/\mathcal{K}$ (see [9]).

This paper is concerned with relationships among the sets $W(T)$, $W(T)^-$, and $W_e(T)$. The following characterization, due to P. Fillmore, J. G. Stampfli, and J. P. Williams [2], is of fundamental importance: $\lambda \in W_e(T)$ if and only if there exists a sequence $\{x_n\}_1^\infty$ of unit vectors in $\mathcal{H}$ such that $x_n \rightarrow 0$ weakly and $(Tx_n, x_n) \rightarrow \lambda$.

In what follows, $\text{Ext}(C)$ denotes the set of extreme points of a convex

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set C , $\text{convh}(X)$ denotes the convex hull of the set X , and ∂X denotes the topological boundary of X .

Theorem 1. *If T is in $\mathcal{B}(\mathcal{H})$, then*

$$\text{Ext}(W(T)^-) \subset W_e(T) \cup W(T).$$

In particular,

$$W(T)^- = \text{convh}(W_e(T) \cup W(T)).$$

Proof. Clearly $W(\alpha T + \beta) = \alpha W(T) + \beta$ for all α, β complex. Therefore, by rotation and translation, we can assume that $W(T)^-$ is contained in the closed right half plane and $0 \in \text{Ext}(W(T)^-) - W(T)$.

Then there exists a sequence $\{x_n\}_1^\infty$ of unit vectors of $\mathcal{H}$ such that $(Tx_n, x_n) \rightarrow 0$. By the weak sequential compactness of the unit ball of $\mathcal{H}$ we can assume that $\{x_n\}_1^\infty$ converges weakly to $x \in \mathcal{H}$ with $\|x\| \leq 1$. We prove that x is the 0 vector, and hence $0 \in W_e(T)$.

If $\|x\| = 1$, then $x_n \rightarrow x$ strongly. But

$$\begin{aligned} |(Tx, x)| &\leq |(T(x - x_n), x)| + |(Tx_n, x - x_n)| + |(Tx_n, x_n)| \\ &\leq \|x - x_n\| \|T^*x\| + \|T\| \|x - x_n\| + |(Tx_n, x_n)| \rightarrow 0. \end{aligned}$$

Hence $(Tx, x) = 0$ and $0 \in W(T)$.

So assume $0 < \|x\| < 1$. Clearly the operator $\text{Re } T$ is positive since $W(T)$ is contained in the closed right half plane. Then

$$\|(\text{Re } T)^{1/2} x_n\|^2 = ((\text{Re } T)x_n, x_n) = \text{Re}(Tx_n, x_n) \rightarrow 0,$$

so $\|(\text{Re } T)x_n\| \rightarrow 0$. This clearly yields $\text{Re}(Tx, x) = 0$, so (Tx, x) is purely imaginary. On the other hand,

$$(T(x - x_n), x - x_n) = (Tx, x - x_n) - (Tx_n, x) + (Tx_n, x_n) \rightarrow -(Tx, x)$$

and

$$\|x - x_n\|^2 = 1 - 2 \text{Re}(x - x_n, x) - \|x\|^2,$$

so $(Ty_n, y_n) \rightarrow -(Tx, x)/(1 - \|x\|^2)$ where $y_n = (x - x_n)/\|x - x_n\|$. Thus we have produced nonzero purely imaginary points in $W(T)^-$ which lie in the upper and lower half planes. However, this implies 0 is a nonextreme point of $W(T)^-$, thus completing the proof of the inclusion. The equality follows from the inclusion by the Krein-Milman theorem.

We remark that an obvious equivalent form of the inclusion in Theorem 1 holds:

$$\text{Ext}(W(T)^-) \subset \text{Ext}(W_e(T)) \cup \text{Ext}(W(T)).$$

Corollary 1. *$W(T)$ is closed if and only if $W_e(T) \subset W(T)$. In particular, a compact operator has closed numerical range if and only if 0 is in its numerical range.*

Proof. Necessity is obvious since $W_e(T) \subset W(T)^-$. If $W_e(T) \subset W(T)$, then $W(T)^- = \text{convh}(W(T)) = W(T)$.

The second statement follows from the first since the essential numerical range of a compact is $\{0\}$.

We remark that the second statement in Corollary 1 is an unpublished observation of A. Brown. It has also been observed by J. R. Giles and B. Sims [3].

Corollary 2. *$W_e(T) = W(T)^-$ if and only if $\text{Ext}(W(T)) \subset W_e(T)$. Therefore, $W_e(T) = W(T)^-$ if $W(T)$ has no extreme points.*

Proof. If $\text{Ext}(W(T)) \subset W_e(T)$, then $\text{Ext}(W(T)^-) \subset W_e(T) \subset W(T)^-$. Taking convex hulls, we obtain $W(T)^- = W_e(T)$. The reverse implication is obvious.

As a specific application of Corollary 2, we mention that $W_e(T_\phi) = W(T_\phi)^-$ for any Toeplitz operator T_ϕ on the Hardy space H^2 (see [4] for definitions). This follows from a theorem of E. Klein [6]: $\text{Ext}(W(T_\phi))$ is void.

For the next result, recall that λ is called a corner of the convex set C if $\lambda \in C$ and C is contained in a sector with vertex λ and opening less than π .

Corollary 3. *If $W(T)^-$ has a corner at the point λ , then λ is in $W_e(T)$ or λ is a reducing eigenvalue of finite multiplicity for T which is an isolated point of $\sigma(T)$.*

Proof. If λ is not in $W_e(T)$, then λ must be a corner of $W(T)$. By a theorem of W. Donoghue [1], λ is an eigenvalue of T . A result of S. Hildebrandt [5] implies that the eigenspace corresponding to λ reduces T . The rest of the proof follows from elementary Fredholm theory. $T - \lambda$ is Fredholm ($\lambda \notin \sigma_e(T)$) of index 0 (λ reduces T); in particular, λ has finite multiplicity. If λ is not isolated, then there exist $\lambda_n \in \sigma(T)$ such that $\lambda_n \neq \lambda$, $\lambda_n \rightarrow \lambda$, and $T - \lambda_n$ is Fredholm of index 0 (this set is open).

Hence, there exist $x_n \in \mathcal{H}$ such that $\|x_n\| = 1$, $x_n \in \operatorname{rg}(T - \lambda)$ and $Tx_n = \lambda_n x_n$. Thus the weak limits of x_n lie in $\operatorname{rg}(T - \lambda) \cap \ker(T - \lambda) = 0$. This yields a contradiction since $x_n \rightarrow 0$ weakly and $(Tx_n, x_n) \rightarrow \lambda$ implies $\lambda \in W_e(T)$.

J. Anderson has communicated the following application of Corollary 3: A closed semidisk cannot be the numerical range of a compact operator. This fact is made evident in the next corollary, which is a mild extension of his idea.

We first give some terminology. Let λ be a corner of $W(T)^-$ and let $B_\epsilon(\lambda) = \{z \in \mathbb{C}: |z - \lambda| < \epsilon\}$. We shall say that λ is *lineal* if there exists $\epsilon > 0$ such that $B_\epsilon(\lambda) \cap \partial W(T)$ consists of line segments emanating from λ .

Corollary 4. *If λ is a corner of $W(T)^-$ and $\lambda \notin W_e(T)$ then λ must be lineal. In particular, if K is compact and λ is a nonzero corner of $W(K)^-$, then λ is lineal.*

Proof. Corollary 3 implies $T = \lambda \oplus T_1$ with respect to $\ker(T - \lambda) \oplus \ker(T - \lambda)^\perp$. Clearly $W(T_1) \subset W(T)$ and $W_e(T_1) \subset W_e(T)$. Hence $\lambda \notin W(T_1)^-$; otherwise, λ would be a corner of $W(T_1)^-$ not in $W_e(T_1)$ and hence an eigenvalue of T_1 , thus producing a nonzero vector in $\ker(T - \lambda) \cap \ker(T - \lambda)^\perp$. Choose $\epsilon > 0$ so that $B_\epsilon(\lambda) \cap W(T_1)^- = \emptyset$. But then (see [4, p. 113])

$$W(T)^- = [\operatorname{convh}(\{\lambda\} \cup W(T_1))]^- = \operatorname{convh}(\{\lambda\} \cup W(T_1)^-).$$

Clearly $B_\epsilon(\lambda) \cap \partial W(T)$ consists of line segments, so λ is lineal. The second statement of the corollary obviously follows from the first.

For the next application recall that T is hyponormal if $T^*T - TT^*$ is positive. T is called *seminormal* if either T or its adjoint T^* is hyponormal.

Corollary 5. *If T is seminormal and λ is an extreme point of $W(T)^-$, then either $\lambda \in W_e(T)$ or λ is a reducing eigenvalue of finite multiplicity for T which is an isolated point of $\sigma(T)$.*

Proof. J. G. Stampfli has proved that extreme points of the numerical range of a seminormal operator are reducing eigenvalues [8]. The proof then proceeds as the one in Corollary 3.

The preceding allows a different proof of a perturbation result due to N. Salinas [7].

Corollary 6. *If T is seminormal, there exists a normal and compact operator K such that T commutes with K and $W(T + K)^- = W_e(T)$.*

Proof. Corollary 5 clearly implies that if T is seminormal,

$$W(T)^- = \text{convh}(W_e(T) \cup \Pi_{\text{oor}}(T)),$$

where $\Pi_{\text{oor}}(T)$ denotes the countable set of isolated reducing eigenvalues of finite multiplicity for T . T can be written as $T = T_1 \oplus \lambda_1 \oplus \lambda_2 \oplus \dots$ with $\{\lambda_n\} = \Pi_{\text{oor}}(T) - W_e(T)$ and T_1 seminormal. If $\{\lambda_n\}$ is finite, choose $\{\mu_n\} \subset W_e(T)$; otherwise, choose $\{\mu_n\} \subset W_e(T)$ so that $|\mu_n - \lambda_n| \rightarrow 0$ (the cluster set of $\{\lambda_n\}$ lies in $\partial W_e(T)$, so $\{\mu_n\}$ exists). In either case, let K be the compact normal operator $0 \oplus (\mu_1 - \lambda_1) \oplus (\mu_2 - \lambda_2) \oplus \dots$. Clearly T commutes with K , $T + K$ is seminormal, and $\Pi_{\text{oor}}(T + K) \subset W_e(T) = W_e(T + K)$. Hence

$$W(T + K)^- = \text{convh}(W_e(T + K) \cup \Pi_{\text{oor}}(T + K)) = \text{convh}(W_e(T)) = W_e(T).$$

The following lemma is a restatement of Lemma 1 in [8].

Lemma. Let $\text{Re } W(T) \geq 0$ and let $\mathcal{J}$ denote the imaginary axis. Then $W(T) \cap \mathcal{J} = \{Tx, x\} : \|x\| = 1 \text{ and } x \in \ker(\text{Re } T)\}$.

An extended version of Theorem 1 can now be obtained.

Theorem 2. For any T in $\mathcal{B}(\mathcal{H})$,

$$W(T)^- = (W_e(T) \cup W(T)) \dot{\cup} \mathcal{L}(T),$$

where $\mathcal{L}(T)$ is a countable disjoint union of open complex line segments each of which has one endpoint in $\text{Ext}(W(T)) - W_e(T)$ and the other in $\text{Ext}(W_e(T)) - W(T)$. ($\dot{\cup}$ denotes disjoint union.)

Proof. Suppose $\lambda \in W(T)^-$ and $\lambda \notin W_e(T) \cup W(T)$. Then by Theorem 1, λ is a nonextreme boundary point of $W(T)^-$. Hence there exists a line l through λ such that $W(T)^-$ lies on one side of l and such that $W(T)^- \cap l = [\alpha, \beta]$ with $\alpha \neq \beta$. Clearly $\alpha, \beta \in \text{Ext}(W(T)^-) \subset W(T) \cup W_e(T)$, so we can assume $\alpha \in \text{Ext}(W(T)) - W_e(T)$ and $\beta \in \text{Ext}(W_e(T)) - W(T)$. But since $W_e(T)$ is closed, $W_e(T) \cap [\alpha, \beta]$ is of the form $[\eta, \beta]$ with $\eta \in \text{Ext}(W_e(T)) - W(T)$. It remains to show that $W(T) \cap [\alpha, \lambda]$ is of the form $[\alpha, \mu]$, so that then $\lambda \in (\mu, \eta) \subset \mathcal{L}(T)$ with μ, η as required.

Without loss of generality we can assume $\text{Re } W(T) \geq 0$, $\eta = 0$, the segment $[\alpha, \lambda]$ lies in the positive imaginary axis with $|\alpha| > |\lambda|$, and $W_e(T) \cap \mathcal{J}$ is contained in the nonpositive imaginary axis. If $\ker(\text{Re } T)$ is infinite dimensional, choose an orthonormal basis $\{e_n\}_1^\infty$ for $\ker(\text{Re } T)$. The lemma then implies $(Te_n, e_n) \in W(T) \cap [\alpha, \lambda]$ for every n . By passing to a

subsequence, if necessary, (Te_n, e_n) converges to a number on the positive imaginary axis. But this number must lie in $W_e(T)$, which is impossible. Hence, $\ker(\operatorname{Re} T)$ must be finite dimensional. Again by the lemma, $W(T) \cap [\alpha, \lambda] = W(T) \cap \mathcal{J} = W(PT|_{\operatorname{rg}(P)})$, where P is the orthogonal projection onto $\ker(\operatorname{Re} T)$ with range $\operatorname{rg}(P)$. Since the numerical range of a finite dimensional operator is closed, $W(T) \cap [\alpha, \lambda]$ is of the desired form.

Corollary 7. *The numerical range of a compact operator contains all its boundary points, with the possible exception of one or two (perhaps degenerate) complex line segments of the form $[0, \alpha]$.*

Examples: The convex set $\operatorname{convh}((0, 1] \cup (i, 2i])$ cannot be the numerical range of a compact operator K because $\partial W(K) - W(K)$ would then be the closed line segment $[0, i]$. It is easy, however, to construct a diagonal compact operator whose numerical range is $\operatorname{convh}((0, 1] \cup [i, 2i])$: let $De_1 = ie_1$, $De_2 = 2ie_2$, and $De_n = (n-2)^{-1}e_n$ for $n \geq 3$ where $\{e_n\}_1^\infty$ is an orthonormal basis for $\mathcal{H}$.

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