A NOTE ON SYMMETRIC FUNCTIONS IN FORMANEK POLYNOMIALS

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ABSTRACT. Formanek's proof of the existence of central identities has the following form: one constructs n polynomials $G(X, Y_1, Y_2, \dots, Y_n)$, $G(X, Y_2, \dots, Y_n, Y_1)$, etc., whose sum is the desired central identity. The variables X, Y_i are generic matrices. These G 's commute pairwise which raises the question whether all symmetric functions in them also give central identities. Here we show that this is not so for $n > 2$, and connect this question with Amitsur's solution of the general crossed product problem.

E. Formanek [1] gave a beautiful solution to a problem of Kaplansky. The problem was to find a polynomial $p(X_1, \dots, X_m)$ in free (noncommuting) variables such that for every substitution $X_i = a_i$, where a_i are $n \times n$ matrices, $p(a_1, \dots, a_m)$ is a central matrix, and not always zero. Such $p(X)$ is called a central identity.

Formanek's ingenious solution was as follows. Let

$$\begin{aligned} g(x_1, \dots, x_{n+1}) &= \prod_{j=2}^n (x_1 - x_j)(x_{n+1} - x_j) \prod_{2 \leq j < k \leq n} (x_j - x_k)^2 \\ &= \sum_{(a)} c_{(a)} x_1^{a_1} x_2^{a_2} \dots x_{n+1}^{a_{n+1}}. \end{aligned}$$

Then let

$$G(X, Y_1, \dots, Y_n) = \sum_{(a)} c_{(a)} X^{a_1} Y_1 X^{a_2} Y_2 \dots X^{a_n} Y_n X^{a_{n+1}}.$$

The sum $G(X, Y_1, \dots, Y_n) + G(X, Y_2, \dots, Y_n, Y_1) + \dots + G(X, Y_1, Y, \dots, Y_{n-1})$ is the desired central identity.

The following question now arises: Are all the basic symmetric functions in the expressions comprising the sum also central identities? Here I show that the answer is no. In §2 of this note I show what the origin of this question was.

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In this note some familiarity with [1] is assumed.

1. For brevity of notation let $G(X, Y_1, \dots, Y_n) = G_1, G(X, Y_2, \dots, Y_1) = G_2$, etc. Let $X = (x_{\alpha\beta}), Y_i = (y_{\alpha\beta}^i)$ where $x_{\alpha\beta}$ and $y_{\alpha\beta}^i$ are independent variables. Here $\alpha, \beta, i = 1, 2, \dots, n$. Such matrices are called generic matrices for obvious reasons. It is easy to see that X is diagonalizable and the basic computation of [1] shows that for X diagonal $G_1, G_2, \dots, G_n$ are also diagonal. Thus $X, G_1, \dots, G_n$ can be simultaneously diagonalized and therefore they commute with each other. Let $s_1, s_2, \dots, s_n$ be the basic symmetric functions. Thus $s_1(x_1, \dots, x_n) = \sum x_i, s_n(x_1, \dots, x_n) = x_1 \cdots x_n$, etc. Suppose $s_j(G_1, \dots, G_n)$ are central identities for $j = 1, 2, \dots, n$. The symmetric functions $p_l(x_1, \dots, x_n) = \sum_{j=1}^n (x_j)^l, l = 1, 2, \dots, n$ can be expressed as integral polynomials of $s_1, \dots, s_n$. Hence $p_l(G_1, \dots, G_n)$ would also be central for $l = 1, 2, \dots, n$. Now we show that this leads to contradiction in two ways, the second of which is less general, but rather illuminating, and is in §2.

To see that $p_j(G_1, \dots, G_n)$ cannot be central for all $j = 1, 2, \dots, n$ it is enough to take X to be diagonal. Thus let $X = (x_i \delta_{ij})$, where $x_1, \dots, x_n$ are indeterminates. As remarked above, for X diagonal all G_i are diagonal. Thus

$$G_j = \begin{pmatrix} b_{j1} & & 0 \\ & \ddots & \\ 0 & & b_{jn} \end{pmatrix}$$

where b_{jk} is a polynomial in the variables $x_i, y_{\alpha\beta}^i$ and $i, j, \alpha, \beta = 1, 2, \dots, n$. The polynomials b_{jk} are all distinct and nonzero. Indeed, one can find a substitution, as in [1], such that $b_{jk} \neq 0$ and $b_{jl} = 0$ for $l \neq k$. Actually it is easy to show that each b_{jk} is the product of $g(x_1, \dots, x_n, x_1)$ with some homogeneous polynomial in the y -s. But this is of no use to us here.

If $\sum_{i=1}^n (G_i)^l$ is a central identity this implies that $\sum_{j=1}^n (b_{jk})^l = H_l$ is constant, i.e. H_l does not depend on k , for all $k = 1, 2, \dots, n$. For example $\sum_{j=1}^n (b_{j1})^l = \sum_{j=1}^n (b_{j2})^l$ for $l = 1, 2, \dots, n$. As no two b_{jk} can be equal for a fixed j this implies that $(b_{12}, \dots, b_{n2})$ is a permutation of $(b_{11}, \dots, b_{n1})$, and in general each $(b_{1k}, \dots, b_{nk})$ is a permutation of $(b_{11}, \dots, b_{n1})$.

Let $Z_i = (z_{\alpha\beta}^i)$ be another set of n generic $n \times n$ matrices. Thus $\{z_{\alpha\beta}^i\}$

is a new set of variables independent of the $x - s$ and $y - s$. Certainly $G_j(X, Z_1, \dots, Z_n)$ is obtained from $G_j(X, Y_1, \dots, Y_n)$ by replacing $y_{\alpha\beta}^i$ by $z_{\alpha\beta}^i$. It follows now that the sum

$$\sum_{j=1}^n G_j(X, Y_1, \dots, Y_n) G_j(X, Z_1, \dots, Z_n)$$

is a central identity. For it is equal to the diagonal matrix with (k, k) th entry which is $\sum_{j=1}^n h_{jk}(x, y) h_{jk}(x, z)$ and this is independent of k as follows from the considerations above. Now we have our contradiction, since the sum $\sum_{i=1}^n G_i(X, Y) G_i(X, Z)$ is not a central identity. Indeed, let $Y_1 = e_{12}$, $Y_2 = e_{23}$, $\dots$, $Y_{n-1} = e_{n-1,n}$, $Y_n = e_{n1}$ and $Z_1 = e_{13}$, $Z_2 = e_{32}$, $Z_3 = e_{24}$, $\dots$, $Z_n = e_{n1}$. Here e_{ij} is the matrix which has 1 in position (i, j) and 0 elsewhere. This substitution gives

$$\begin{aligned} G_1(X, Y) &= g(x) e_{11}, & G_2(X, Y) &= g(x) e_{22}, \dots, & G_n(X, Y) &= g(x) e_{nn}, \\ G_1(X, Z) &= g(x) e_{11}, & G_2(X, Z) &= g(x) e_{33}, \dots, & G_n(X, Z) &= g(x) e_{nn}. \end{aligned}$$

Thus $\sum_{j=1}^n G_j(X, Y) G_j(X, Z)$ is not 0 but has 0 in the $(2, 2)$ position and therefore is not central.

To sum up this section we make a formal statement.

Theorem. Let $s_1, \dots, s_n$ be the basic symmetric polynomials. Then $s_1(G_1, \dots, G_n), \dots, s_n(G_1, \dots, G_n)$ are not all central identities.

2. Amitsur [2] proved that the following division algebra is not a crossed product. Let n be an integer divisible by p^2 , where p is an odd prime, or by 8. Let F be a field of characteristic 0, and let $X_1, \dots, X_m$, $m \geq 2$, be generic $n \times n$ matrices. Then the ring $R = F[X_1, \dots, X_m]$ is a domain and it has a total ring of fractions D which is a division ring of dimension n over its center K . This D was Amitsur's example. In our case $R = F[X, Y_1, \dots, Y_n]$ with D, K having the analogous meaning.

Now suppose the Theorem proved is false. This would mean that the polynomial $p(t) = (t - G_1)(t - G_2) \dots (t - G_n)$ has coefficients in K . Furthermore, it is irreducible because it is clear that no sum $G_{i_1} + \dots + G_{i_r}$, where $(i_1, \dots, i_r)$ is a proper nonempty subset of $(1, 2, \dots, n)$, is a central identity. Thus the field $K(G_1, \dots, G_n)$ is a Galois extension, even a cyclic extension of K , of degree n over K and hence a maximal subfield of D . This implies that D is cyclic, which is absurd.

We end by remarking that perhaps if n is prime one might find a polynomial, using Formanek's trick, such that the symmetric functions are central. This, of course, will resolve the part of the crossed product problem still open.

REFERENCES

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