LOCALIZATIONS AND EVALUATION SUBGROUPS

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ABSTRACT. If $G_n(X)$ is the n th evaluation subgroup of a simple connected finite CW-complex, then $G_n(X_p) \cong G_n(X)_p$ for $p = 0$ or a prime.

Let X be a connected simple finite CW-complex, X_p its localization at p for p prime or 0 [5], and $G_n(X)$ the evaluation subgroup of $\pi_n(X)$ [1]. If $e_p: X \rightarrow X_p$ is the canonical map, we show that $\alpha \in G_n(X)$ if and only if $e_{p*}(\alpha) \in G_n(X_p)$ for all p . As corollaries we obtain that $G_n(X_p) \cong G_n(X_p)$ and X is a G -space [1] if and only if X_p is a G -space for all p , where $G_n(X)_p$ is the localization of the group $G_n(X)$. This is analogous to results obtained for H -spaces [3], [5]. All H -spaces are G -spaces, and many properties of H -spaces are shared by G -spaces.

1. Preliminaries. Spaces X and W are assumed to be pointed, simple (abelian fundamental groups acting trivially on the homotopy and homology groups), connected, finite CW-complexes. We will not distinguish between a map and its homotopy class. For p a prime let $\mathbb{Q}_p = \{k/q | k, q \text{ integers, } p \nmid q\}$ and $\mathbb{Q}_0$ the rationals. $\mathbb{Q}_p$ is the localization of the integers at the prime p . The general reference for localization theory is [5]. We review some of these results here.

Definition 1.1. A space X is p -local if $\pi_*(X)$ admits a $\mathbb{Q}_p$ -module structure extending the usual $\mathbb{Z}$ -module structure.

For each X there is a p -local space X_p and canonical map $e_p: X \rightarrow X_p$ such that if $g: X \rightarrow Y$, where Y is p -local, there is a unique (up to homotopy) $g': X_p \rightarrow Y$ such that $g \simeq g' e_p$. This is equivalent to the map $\phi_p: \pi_n(X) \otimes \mathbb{Q}_p \rightarrow \pi_n(X_p)$ being an isomorphism, where $\phi_p(\alpha \otimes r) = r e_{p*}(\alpha)$, where the multiplication is the $\mathbb{Q}_p$ -module structure on $\pi_n(X_p)$ (e_p p -localizes in homotopy). Finally we point out that localization is functorial.

Key results about the evaluation subgroups can be found in [1]. We will establish the notations needed in this paper. $L(W, X; f)$ will be the

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path component containing $f: W \rightarrow X$ in the space of functions from W to X with the compact-open topology; $L_0(W, X; f)$ will be the subspace of $L(W, X; f)$ containing base point preserving functions; $\omega: L(W, X; f) \rightarrow X$ will be the function which evaluates each map at the base point of W .

Definition 1.2. The evaluation subgroup, $G_n(X)$, of $\pi_n(X)$ is defined by

$$G_n(X) = \text{Im}(\omega_*: \pi_n(L(X, X; 1_X)) \rightarrow \pi_n(X))$$

where 1_X is the identity on X .

$G_n(X)$ consists of all elements $\alpha \in \pi_n(X)$ such that there is a function $F: X \times S^n \rightarrow X$ with $F|_{X \vee S^n} = 1_X \vee \alpha$. Such a function will be called an affiliated map for α .

2. Localizations and evaluation subgroups. Let

$$(1) \quad \hat{e}_p: L(W, X; f) \rightarrow L(W, X_p; e_p f)$$

be defined by $\hat{e}_p(g) = e_p g$.

Theorem 2.1. If W is a connected finite CW-complex and X a connected simple CW-complex then (1) p -localizes in homotopy.

Proof. By Corollary 1.3 in [3] $\bar{e}_p = \hat{e}_p|_{L_0(W, X; f)}$ p -localizes. Consider the following commutative diagram:

$$\begin{array}{ccc} L_0(W, X; f) & \xrightarrow{\bar{e}_p} & L_0(W, X_p; e_p f) \\ \downarrow & & \downarrow \\ L(W, X; f) & \xrightarrow{\hat{e}_p} & L(W, X_p; e_p f) \\ \downarrow \omega & & \downarrow \omega \\ X & \xrightarrow{e_p} & X_p \end{array}$$

Since ω is a fibration in both columns and two of the three horizontal maps p -localize, $\hat{e}_p$ also p -localizes (see 2.21 and [5]).

Theorem 2.2. If $\alpha \in G_n(X)$ then $e_p * (\alpha) \in G_n(X_p)$.

Proof. Let $F: X \times S^n \rightarrow X$ be an affiliated map for α . Let $\bar{F}$ be the composition

$$X_p \times S^n \xrightarrow{1 \times e_p} X_p \times S_p^n \xrightarrow{F_p} X_p.$$

Note that $(X \times S^n)_p = X_p \times S_p^n$. Since $\bar{F}|_{S^n} = \alpha_p e_p = e_p \alpha$, $\bar{F}$ is an affiliated map for $e_p * (\alpha)$. This result does not require X to be finite.

Theorem 2.3. If $e_p * (\alpha) \in G_n(X_p)$ for all p then $\alpha \in G_n(X)$.

Proof. *Case 1.* α of finite order. Let $p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}$ be the prime factorization of the order α . Let $P_1 = \prod_{i \neq 1} p_i^{m_i}$. Since $p_1^{m_1}$ and P_1 are relatively prime, there are integers r and s such that $1 = rp_1^{m_1} + sP_1$; so $\alpha = rp_1^{m_1}\alpha + sP_1\alpha$. $sP_1\alpha$ is of order $p_1^{m_1}$ while $rp_1^{m_1}\alpha$ is of order P_1 . $rp_1^{m_1}\alpha$ can then be written as a sum of its (hence α 's) multiples such that one summand is of order $p_2^{m_2}$ and the other of order $\prod_{i > 2} p_i^{m_i}$. By induction $\alpha = k_1\alpha + k_2\alpha + \cdots + k_n\alpha$ where $k_i\alpha$ is of order $p_i^{m_i}$. Since $e_{p^*}(\alpha) \in G_n(X_p)$ for all p , $e_{p^*}(k_i\alpha) \in G_n(X_p)$. So if we can show the result for α of order p^m , p any prime, then each $k_i\alpha$ will be in $G_n(X)$ and thus $\alpha \in G_n(X)$.

We assume then that α has order p^m for p prime and localize at this p . Consider the following diagram:

$$\begin{array}{ccc} \pi_n(L(X, X_p; e_p)) & \xrightarrow{\omega_*} & \pi_n(X_p) \\ \uparrow \bar{e}_{p^*} & \nearrow \omega_* & \\ \pi_n(L(X_p, X_p; 1)) & & \end{array}$$

where $\bar{e}_p(f) = fe_p$. From this commutative diagram and the definition of $e_{p^*}(\alpha) \in G_n(X_p)$ it follows that there is an $\tilde{\alpha} \in \pi_n(L(X, X_p; e_p))$ such that $\omega_*(\tilde{\alpha}) = e_{p^*}(\alpha)$. Let

$$\bar{\phi}_p: \pi_n(L(X, X; 1_X)) \otimes Q_p \rightarrow \pi_n(L(X, X_p; e_p))$$

be the isomorphism guaranteed by Theorem 2.1. There is an element

$$\sum (\alpha_i \otimes (k_i/q_i)) \in \pi_n(L(X, X; 1_X)) \otimes Q_p$$

such that $\bar{\phi}_p(\sum (\alpha_i \otimes (k_i/q_i))) = \tilde{\alpha}$. If $q = \prod q_i$ and $\bar{q}_i = \prod_{j \neq i} q_j$ then

$$\begin{aligned} \sum (\alpha_i \otimes (k_i/q_i)) &= \sum (\alpha_i \otimes \bar{q}_i k_i/q) \\ &= \sum (\bar{q}_i k_i \alpha_i \otimes (1/q)) = \left(\sum \bar{q}_i k_i \alpha_i \right) \otimes (1/q). \end{aligned}$$

Note that $p \nmid q_i$ for each i , hence $p \nmid q$. Thus there is an element

$$\beta \otimes (1/q) \in \pi_n(L(X, X; 1_X)) \otimes Q_p$$

such that $\bar{\phi}_p(\beta \otimes (1/q)) = \tilde{\alpha}$ and $p \nmid q$. Consider the following diagram:

$$\begin{array}{ccccc}
 \pi_n(L(X, X; 1_X)) \otimes \mathcal{Q}_p & \xrightarrow{\omega_* \otimes 1} & \pi_n(X) \otimes \mathcal{Q}_p & \xrightarrow{\phi_p} & \pi_n(X_p) \\
 & \searrow \bar{\phi}_p & & \nearrow \omega_* & \\
 & & \pi_n(L(X, X_p; e_p)) & &
 \end{array}$$

By commutativity

$$\begin{aligned}
 \phi_p(\omega_*(\beta) \otimes 1) &= \omega_*(\bar{\phi}_p(\beta \otimes 1)) = \omega_*(\bar{\phi}_p(q\beta \otimes (1/q))) \\
 &= \omega_*(q\alpha) = e_{p*}(q\tilde{\alpha}).
 \end{aligned}$$

But $e_{p*}(q\alpha) = \phi_p(q\alpha \otimes 1)$, so $\omega_*(\beta) \otimes 1 = q\alpha \otimes 1$ since ϕ_p is an isomorphism. But then $\omega_*(\beta) = q\alpha + \gamma$ where γ has order q' and $p \nmid q'$, so $\omega_*(q'\beta) = q'q\alpha$. Since p^m and $q'q$ are relatively prime there are integers r and s such that $rp^m + sq'q = 1$. Thus $\omega_*(sq'\beta) = sq'q\alpha = \alpha - rp^m\alpha = \alpha$ and $\alpha \in G_n(X)$.

Case 2. α of infinite order. Localizing at 0 the above argument yields an element

$$\beta \otimes (1/q) \in \pi_n(L(X, X; 1_X)) \otimes \mathcal{Q}_0$$

such that $\omega_*(\beta) \otimes (1/q) = \alpha \otimes 1$, and as above $\omega_*(q'\beta) = q'q\alpha$ for some nonzero integer q' . Thus there are nonzero multiples of α in $G_n(X)$. Let $\bar{q}$ be the least positive integer such that $\bar{q}\alpha \in G_n(X)$. If $\bar{q} \neq 1$ let p be a prime factor of $\bar{q}$. Localizing at this p we obtain an element $\beta' \in \pi_n(L(X, X; 1_X))$ and a q'' where $p \nmid q''$ such that $\omega_*(\beta') = q''\alpha$. But then $q''\alpha$ is in the subgroup generated by $\bar{q}\alpha$ so q'' is a multiple of $\bar{q}$. This is a contradiction since p is a factor of $\bar{q}$ but not of q'' . Thus $\bar{q} = 1$ and $\alpha \in G_n(X)$.

During the proof we also obtained the following useful corollary.

Corollary 2.4. *If $e_{p*}(\alpha) \in G_n(X_p)$ then there is a q such that $p \nmid q$ and $q\alpha \in G_n(X)$.*

Corollary 2.5. $G_n(X_p) \cong G_n(X)_p$.

Proof. We need only show that $\phi_p: \pi_n(X) \otimes \mathcal{Q}_p \rightarrow \pi_n(X_p)$ satisfies

$$\text{Im } \phi_p | G_n(X) \otimes \mathcal{Q}_p = G_n(X_p).$$

Say $\alpha \otimes (1/q) \in G_n(X) \otimes \mathcal{Q}_p$ (we can assume any element in $G_n(X) \otimes \mathcal{Q}_p$ has this form by the technique used in 2.3). Let $F: X \times S^n \rightarrow X$ be an

affiliated map for α and $g: S^n \rightarrow S_p^n$ a map corresponding to $1/q \in \pi_n(S_p^n) \cong Q_p$. Then the composition

$$X_p \times S^n \xrightarrow{1 \times g} X_p \times S_p^n \xrightarrow{F_p} X_p$$

is an affiliated map for $\phi_p(\alpha \otimes (1/q))$ and

$$\text{Im } \phi_p|_{G_n(X) \otimes Q_p} \subset G_n(X_p).$$

Now say $\tilde{\alpha} \in G_n(X_p)$ and $\alpha \otimes (1/q) = \phi_p^{-1}(\tilde{\alpha})$. Then $e_{p*}(\alpha) \otimes 1 = q\tilde{\alpha} \otimes 1$ and $e_{p*}(\alpha) - q\tilde{\alpha}$ is of order q' where $p \nmid q'$. Thus $e_{p*}(q'\alpha) = q'q\tilde{\alpha} \in G_n(X_p)$. By Corollary 2.5 there is a q'' such that $p \nmid q''$ and $q''q'\alpha \in G_n(X)$. Then

$$q''q'\alpha \otimes \frac{1}{q''q'q} \in G_n(X) \otimes Q_p \quad \text{and} \quad \phi_p\left(q''q'\alpha \otimes \frac{1}{q''q'q}\right) = \phi_p\left(\alpha \otimes \frac{1}{q}\right) = \tilde{\alpha},$$

so $\tilde{\alpha} \in \text{Im } \phi_p|_{G_n(X) \otimes Q_p}$. Thus $\phi_p|_{G_n(X) \otimes Q_p}$ is onto $G_n(X_p)$.

Corollary 2.6. X is a G -space if and only if X_p is a G -space for all p .

Proof. By definition X is a G -space if and only if $G_n(X) = \pi_n(X)$ for all n . If X is a G -space, X_p is a G -space for all p by Corollary 2.5. If X_p is a G -space for all p then X is a G -space by Theorem 2.3.

The following application illustrates how these results can be used.

Corollary 2.7. If $G_n(X)$ is torsion without p -torsion, any fibration $f: E \rightarrow S^{n+1}$ with fiber X_p admits a cross section.

Proof. In this case $G_n(X)_p = 0$ so by Corollary 2.5 $G_n(X_p) = 0$ and the result follows from Corollary 2.7 in [1].

For example, if V is the real Stiefel manifold $V_{n,k}$ where $n-k=1$ or 3 , $k>1$, any fibration over S^{n-k+1} with fiber V_p , p an odd prime, admits a cross section. This follows from Corollary 2.7 and results in [4].

Corollary 2.4 also provides a shortened proof of Proposition 3 in [2] with simply connected replaced by simple. Let $\mathcal{C}$ be a Serre class of Abelian groups. Haslam defines a G -space mod $\mathcal{C}$ to be a space X such that $\pi_n(X)/G_n(X) \in \mathcal{C}$. A space will be called an H -space mod $\mathcal{C}$ if there is a map $\mu: X \times X \rightarrow X$ which, when restricted to each factor, induces a $\mathcal{C}$ isomorphism in homotopy. Let $\mathcal{C}_p$ be the class of finite Abelian groups having no element with order a positive power of p ; let $\mathcal{C}_0 = \mathcal{F}$ be the class of finite Abelian groups.

Theorem 2.8 (Haslam). *If X is an H -space mod $\mathcal{C}_p$, p prime or 0, then X is a G -space mod $\mathcal{C}_p$.*

Proof. If X is an H -space mod $\mathcal{C}_p$ then it can be shown that X_p is an H -space and hence a G -space. For every $\alpha \in \pi_n(X)$, $e_p*(\alpha) \in G_n(X_p)$ and by Corollary 2.4 there is a q where $p \nmid q$ such that $q\alpha \in G_n(X)$. Thus X is a G -space mod $\mathcal{C}_p$.

For $\mathcal{C}_0$ Haslam has obtained a converse to this theorem [2]. For $\mathcal{C}_p$, p prime, the converse is false in general but the simply connected case is still open.

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