EXISTENCE AND CONJUGACY OF CARTAN SUBALGEBRAS OF JORDAN ALGEBRAS

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ABSTRACT. It is shown that a finite-dimensional Jordan algebra over a field k of characteristic $\neq 2$ contains Cartan subalgebras, and that any two Cartan subalgebras are conjugate by an inner automorphism provided k is algebraically closed.

Let $\mathfrak{J}$ be a finite-dimensional unital (linear) Jordan algebra over a field k of characteristic $\neq 2$. The purpose of this note is to complete N. Jacobson's theory of Cartan subalgebras (cf. [3]) by proving

- (i) $\mathfrak{J}$ contains Cartan subalgebras;
- (ii) if k is algebraically closed then any two Cartan subalgebras are conjugate by an inner automorphism of $\mathfrak{J}$;
- (iii) if k is infinite then every Cartan subalgebra contains associator regular elements.

The proof of (ii) is essentially the same as Jacobson's, once we know that the Lie algebra of the inner automorphism group contains all inner derivations. The existence of Cartan subalgebras was proved for infinite fields in [3] and for finite fields with sufficiently many elements in [2]. Our proof of (i) uses (ii) and imitates the corresponding proof for the existence of Cartan subgroups of algebraic groups [1]. These methods seem to break down for quadratic Jordan algebras over fields of characteristic 2, mainly due to the fact that the inner automorphism group is no longer smooth (in the sense of group schemes). However, in the broader framework of Jordan pairs, there is a satisfactory theory of Cartan subpairs over fields of arbitrary characteristic (cf. [4]).

We denote by $xy = R_x(y)$ the multiplication in $\mathfrak{J}$, by $U_x = 2R_x^2 - R_{x^2}$ the quadratic representation, and let

$$U_{x,y} = U_{x+y} - U_x - U_y = 2(R_x R_y + R_y R_x - R_{xy}).$$

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Throughout, K will be an algebraic closure of k and k_s the separable closure of k in K . We imbed $\mathfrak{J}$ into $\mathfrak{J}_K = \mathfrak{J} \otimes_k K$ via $x \mapsto x \otimes 1$. Our reference for algebraic groups is [1]. The Lie algebra (resp. connected component of the identity) of an algebraic group G is denoted $L(G)$ or $\mathfrak{g}$ (resp. G^0). Recall that $L(G) = L(G^0)$ and G^0 is defined over k provided G is (see [1, pp. 86, 118]).

The *inner structure group* of $\mathfrak{J}$ is the subgroup G of $\text{GL}(\mathfrak{J}_K)$ generated by all U_x , $x \in \mathfrak{J}_K$ invertible. By [1, p. 106], G is a connected algebraic group, defined over k . Its Lie algebra $\mathfrak{g} = L(G)$ contains all R_y , $y \in \mathfrak{J}_K$, since $2R_y$ is the derivative of U_x at $x = 1$ in direction y . For $g \in G$ let

$$(1) \quad \theta(g) = U_{g(1)}^{-1} g.$$

Then θ is an automorphism of period 2 of G which is defined over k , and we have

$$(2) \quad U_{g(x)} = g U_x \theta(g)^{-1},$$

$$(3) \quad \theta(U_x) = U_x^{-1},$$

for all invertible $x \in \mathfrak{J}_K$. Let G_θ be the fixed point set of θ in G , let $L(\theta)$ be the automorphism of $\mathfrak{g}$ induced by θ , and let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be the decomposition of $\mathfrak{g}$ into the (± 1) -eigenspaces of $L(\theta)$. Clearly $\mathfrak{g}_+$ is a subalgebra of $\mathfrak{g}$, and we have $[\mathfrak{g}_+, \mathfrak{g}] \subset \mathfrak{g}_-$ and $[\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+$. From (3) we get by differentiating

$$(4) \quad L(\theta)(R_x) = -R_x \in \mathfrak{g}_-,$$

and hence

$$(5) \quad D_{x,y} = [R_x, R_y] \in \mathfrak{g}_+,$$

for all $x, y \in \mathfrak{J}_K$. The $D_{x,y}$ are called *inner derivations*.

Lemma 1. G_θ is defined over k , and $L(G_\theta) = \mathfrak{g}_+$.

Proof. Let $A = G \times (\mathbb{Z}/2\mathbb{Z})$ be the semidirect product with the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acting on G via θ . Then A is an algebraic group defined over k , and $s = (e, \theta)$ is a k -rational point of A which is moreover semi-simple since it is of order two and $\text{char}(k) \neq 2$. By [1, p. 225], the centralizer $Z_G(s)$ of s in G is defined over k , and its Lie algebra is the centralizer $Z_{\mathfrak{g}}(s)$ of s in $\mathfrak{g}$. From the definition it is clear that $Z_G(s) = G_\theta$ and $Z_{\mathfrak{g}}(s) = \mathfrak{g}_+$.

The *inner automorphism group* of $\mathfrak{J}$ is defined by $H = \{g \in G: g(1) = 1\}$. Clearly, H is k -closed, and by (1) it is contained in G_θ .

Lemma 2. *H is of finite index in G_θ , and hence we have $H^0 = (G_\theta)^0$ and $L(H) = \mathfrak{g}_+$. In particular, $L(H)$ contains all inner derivations. Moreover, H is defined over k .*

Proof. Let $X = \{x \in \mathfrak{Y}_K : U_x = \text{Id}\}$. Then X is an algebraic variety, G_θ acts on X by (2), and H is the isotropy group of 1 in G_θ . We show that X is finite by proving that the tangent space $T_x(X)$ is zero, for every $x \in X$. Indeed, $y \in T_x(X)$ if and only if $U_{x,y} = 0$, and this implies $0 = U_{x,y}(x) = 2x^2y = 2y$ since $x^2 = U_x(1) = 1$. By [1, p. 86] we have $(G_\theta)^0 \subset H$ which proves that $H^0 = (G_\theta)^0$ is defined over k and $L(H) = L(G_\theta) = \mathfrak{g}_+$ by Lemma 1. Since H is obviously stable under the Galois group of k_s over k it follows that H is defined over k [1, p. 57].

Lemma 3. *Suppose $k = K$ is algebraically closed. Let $\mathfrak{C}$ be a subalgebra of $\mathfrak{Y}$ containing $\sum_{i=1}^r \mathfrak{Y}_{ii}$ where $\mathfrak{Y} = \sum \mathfrak{Y}_{ij}$ is the Peirce decomposition of $\mathfrak{Y}$ with respect to a supplementary set $\{e_1, \dots, e_r\}$ of orthogonal idempotents. Then the orbit map $\alpha: H^0 \times \mathfrak{C} \rightarrow \mathfrak{Y}$ is dominant and separable, and hence the orbit of $\mathfrak{C}$ under H^0 contains an open dense subset of $\mathfrak{Y}$ (in the Zariski topology).*

Proof. By [1, p. 75] it suffices to show that the differential of α is surjective at some point of $H^0 \times \mathfrak{C}$. Let $x = \sum \lambda_i e_i$ where the $\lambda_i \in k$ are such that $\lambda_i \neq \lambda_j$ for $i \neq j$. By Lemma 2, the tangent space of $H^0 \times \mathfrak{C}$ at $a = (\text{Id}, x)$ is $\mathfrak{g}_+ \times \mathfrak{C}$, and $(d\alpha)_a$ is given by

$$(d\alpha)_a(D, y) = D(x) + y,$$

for $(D, y) \in \mathfrak{g}_+ \times \mathfrak{C}$. Hence we only have to show that all $\mathfrak{Y}_{ij}$ for $i \neq j$ are contained in the image of $(d\alpha)_a$. But if $z \in \mathfrak{Y}_{ij}$ then by Lemma 2, $D_{z,x} \in \mathfrak{g}_+$, and an easy computation using the Peirce rules shows

$$D_{z,x}(x) = zx^2 - (zx)x = \frac{1}{4}(\lambda_i - \lambda_j)^2 z.$$

This proves the lemma.

Let $[x \ y \ z] = (xy)z - x(yz) = R_{x,y}(z)$ denote the associator in $\mathfrak{Y}$ (resp. $\mathfrak{Y}_K$). For $x \in \mathfrak{Y}_K$ let $T^n + c_1(x)T^{n-1} + \dots + c_n(x)$ be the characteristic polynomial of $R_{x,x}$. Then the c_i are polynomial functions on $\mathfrak{Y}_K$ which are defined over k . Let m be the largest integer such that $c_{n-i} = 0$ for $i < m$. An element x is called (associator) *regular* if $c_{n-m}(x) \neq 0$. (If k is finite it may happen that $c_{n-m}(x) = 0$ for all $x \in \mathfrak{Y}$.) The set $\mathfrak{Y}_{\text{reg}}$ of regular elements of $\mathfrak{Y}_K$ is open and dense.

Recall that a Jordan algebra is *associator nilpotent* if there exists an

integer n such that any product of n of the $R_{x,y}$'s vanishes. A subalgebra $\mathbb{C}$ of $\mathfrak{J}$ is called a *Cartan subalgebra* if it is associator nilpotent and equal to its own normalizer in the sense that $[\mathbb{C} \mathbb{C} z] \subset \mathbb{C}$ implies $z \in \mathbb{C}$. If x is regular then

$$\mathfrak{Z}_x = \{y \in \mathfrak{J} : R_{x,x}^n(y) = 0 \text{ for some } n\}$$

is the unique Cartan subalgebra of $\mathfrak{J}$ containing x . If k is algebraically closed then $\mathbb{C} = \sum \mathfrak{J}_{ii}$ with respect to a supplementary set of orthogonal primitive idempotents (cf. [3, pp. 343ff]).

Theorem 1. *If k is infinite then every Cartan subalgebra $\mathbb{C}$ of $\mathfrak{J}$ contains regular elements and hence is of the form $\mathfrak{Z}_x$ with x regular.*

Proof. First let k be algebraically closed and $\mathbb{C} = \sum \mathfrak{J}_{ii}$ as above. By Lemma 3, $H^0 \cdot \mathbb{C}$ contains a dense open subset of $\mathfrak{J}$. Since $\mathfrak{J}_{\text{reg}}$ is invariant under all automorphisms of $\mathfrak{J}$ we have $\emptyset \neq (H^0 \cdot \mathbb{C}) \cap \mathfrak{J}_{\text{reg}} = H^0 \cdot (\mathbb{C} \cap \mathfrak{J}_{\text{reg}})$, and $\mathbb{C}$ contains regular elements. If k is not algebraically closed then $\mathbb{C}_K$ is a Cartan subalgebra of $\mathfrak{J}_K$ and therefore contains regular elements. Thus the function c_{n-m} does not vanish on $\mathbb{C}_K$, and since k is infinite it does not vanish on $\mathbb{C}$.

Theorem 2. *If k is algebraically closed then any two Cartan subalgebras of $\mathfrak{J}$ are conjugate by an element of H^0 .*

Proof. Let $\mathbb{C}$ and $\mathbb{C}'$ be Cartan subalgebras of $\mathfrak{J}$. By Lemma 3,

$$\emptyset \neq (H^0 \cdot \mathbb{C}) \cap (H^0 \cdot \mathbb{C}') \cap \mathfrak{J}_{\text{reg}} = H^0 \cdot (\mathbb{C} \cap (H^0 \cdot \mathbb{C}') \cap \mathfrak{J}_{\text{reg}}).$$

Thus there exists a regular element $x \in \mathbb{C}'$ and $h \in H^0$ such that $h(x) \in \mathbb{C}$. It follows that $h(\mathbb{C}') = h(\mathfrak{Z}_x) = \mathfrak{Z}_{h(x)} = \mathbb{C}$.

Theorem 3. *Let k be arbitrary. Then $\mathfrak{J}$ contains a Cartan subalgebra.*

Proof. If k is infinite then $\mathfrak{J}$ contains regular elements and therefore Cartan subalgebras. Let k be finite and let q be the number of elements of k . Then the Galois group of $K = k_s$ over k is generated (topologically) by the Frobenius map $F: \lambda \mapsto \lambda^q$. If X is an algebraic variety defined over k we denote by $x \mapsto x^{(q)}$ the action of F on X . Since H^0 is defined over k (Lemma 2) the map $g \mapsto g^{-1}g^{(q)}$ from H^0 into itself is surjective [1, p. 369]. The action of F on $\mathfrak{J}_K$ obviously satisfies

$$(x + y)^{(q)} = x^{(q)} + y^{(q)}, \quad (\lambda x)^{(q)} = \lambda^q x^{(q)}, \quad \text{and} \quad (xy)^{(q)} = x^{(q)} y^{(q)}.$$

Hence if $\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{J}_K$ so is $\mathbb{C}^{(q)}$. By Theorem 2 there

exists $g \in H^0$ such that $g(\mathbb{C}^{(q)}) = \mathbb{C}$, and $g = h^{-1}h^{(q)}$ for some $h \in H^0$. This implies $h(\mathbb{C}) = h^{(q)}(\mathbb{C}^{(q)}) = h(\mathbb{C})^{(q)}$ and therefore $h(\mathbb{C})$ is defined over k [1, pp. 52ff]. Now $h(\mathbb{C}) \cap \mathfrak{J}$ is a Cartan subalgebra of $\mathfrak{J}$.

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