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HOMOLOGICAL ALGEBRA IN TOPOI

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ABSTRACT. Let $\mathbf{A}$ be the category of sheaves of universal algebras of some fixed type on a Grothendieck topology. There is defined a cohomology theory H^* such that if A is an object in $\mathbf{A}$ and M is an $\mathbf{A}$ -module then $H^1(A, M)$ is in one-to-one correspondence with the equivalence classes of singular extensions of A by M . When $\mathbf{A}$ is the category of sheaves of R -modules for some sheaf of rings R , then $H^n(A, M) \cong \text{Ext}^n(A, M)$ for all $n \geq 0$.

The purpose of this note is to extend the results of [4] and [5] to general topoi, i.e. to sheaves on a Grothendieck topology. The extension has been made possible by the following theorem: Let $\mathbf{E}$ be a topos. Then there exists a complete Boolean algebra B and a left exact cotriple G on $\tilde{\mathbf{B}}$ (= sheaves on B for the canonical topology) such that $\mathbf{E}$ and $\tilde{\mathbf{B}}_G$ (= the category of G -coalgebras) are equivalent categories.

Barr's proof of this theorem can be found in [1]. Lawvere, who conjectured the theorem, paraphrases it by saying that any topos has enough Boolean-valued points. The value of the theorem is that in $\tilde{\mathbf{B}}$ subobjects have complements and epimorphisms split, i.e. $\tilde{\mathbf{B}}$ is very "set-like". One

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uses this either indirectly or directly by transferring questions about $\mathbf{E}$ to questions about $\tilde{\mathbf{B}}$. In this note we use the former approach.

Let $\mathbf{Th}$ be a finitary theory (see, for example, [3]), and let $\text{Alg}(\mathbf{Th}, \mathbf{E})$ be the category of product preserving functors $\mathbf{Th} \rightarrow \mathbf{E}$. For example, when $\mathbf{Th}$ is the theory of abelian groups, $\text{Alg}(\mathbf{Th}, \mathbf{E})$ is the category of abelian group objects in $\mathbf{E}$. Let

$$\mathbf{E} \begin{matrix} \xrightarrow{S} \\ \xleftarrow{Q} \end{matrix} \tilde{\mathbf{B}}$$

be the adjoint pair arising from the Barr-Lawvere theorem, $U: \text{Alg}(\mathbf{Th}, \tilde{\mathbf{B}}) \rightarrow \tilde{\mathbf{B}}$ and $U': \text{Alg}(\mathbf{Th}, \mathbf{E}) \rightarrow \mathbf{E}$ the usual "forgetful" functors. Since both S and Q preserve finite products, we get functors

$$\text{Alg}(\mathbf{Th}, \mathbf{E}) \xrightarrow{S'} \text{Alg}(\mathbf{Th}, \tilde{\mathbf{B}}) \xrightarrow{Q'} \text{Alg}(\mathbf{Th}, \mathbf{E})$$

via composition, e.g. given $A: \mathbf{Th} \rightarrow \mathbf{E}$ we get $S'(A) = \mathbf{Th} \xrightarrow{A} \mathbf{E} \xrightarrow{S} \tilde{\mathbf{B}}$. An easy computation shows that S' is left adjoint to Q' , and in fact that the cotriple $S'Q'$ is a lifting of $SQ = G$ to $\text{Alg}(\mathbf{Th}, \tilde{\mathbf{B}})$ in the sense of [6]. Moreover U is tripleable (see [7] for the first proof, but a direct proof is not difficult to give).

Hence there is a mixed distributive law $\lambda: UFSQ \rightarrow SQUF$ where F is left adjoint to U (see [6, pp. 235–236]) and a lifting of the triple UF to a triple T' on $\mathbf{E}$. Using the fact that $US' = SU'$ one can show that S' is cotripleable and U' is tripleable. It follows that $\text{Alg}(\mathbf{Th}, \mathbf{E})$ is bialgebraic over $\tilde{\mathbf{B}}$ in the sense of [4], and hence that there is a cohomology theory, the first group of which classifies principal objects in $\text{Alg}(\mathbf{Th}, \mathbf{E})$. For example, let $\mathbf{Th}$ be the theory of commutative rings, let R be a $\mathbf{Th}$ -algebra (= a sheaf of commutative rings), let M be an R -module (= a sheaf of R -modules) and

$$C^{i,j}(R, M) = \text{Der}_R((F'U')^{i+1}R, (Q'S')^{j+1}M).$$

Then the first homology group of the double complex $C^{*,*}(R, M)$ is in one-to-one correspondence with the set of equivalence classes of singular extensions of R by M . The "local triviality" condition which must be met is guaranteed because epimorphisms split in $\tilde{\mathbf{B}}$. Notice that this provides a theoretical solution to the problem posed by Grothendieck in [2]. See [4] for more details and more examples.

In case that $\mathbf{Th} = \mathbf{Ab}$ is the theory of abelian groups we can classify n -fold extensions for each $n \geq 1$. Let M, N be objects in $\text{Alg}(\mathbf{Ab}, \mathbf{E})$, let G' be the free abelian group cotriple lifted to $\text{Alg}(\mathbf{Ab}, \mathbf{E})$, and $T' = Q'S'$ as above. Let

$$C^{i,j} = \text{Alg}(\mathbf{Ab}, \mathbf{E}) (G'^{i+1}M, T'^{j+1}N);$$

then the n th homology group of $C^{*,*}$ is in one-to-one correspondence with $\text{Ext}_R^n(M, N)$, the set of equivalence classes of exact sequences $0 \rightarrow N \rightarrow A_0 \rightarrow \cdots \rightarrow A_{n-1} \rightarrow M \rightarrow 0$ in $\text{Alg}(\mathbf{Ab}, \mathbf{E})$. The proof of this fact is identical with that given in [5]. The argument rests ultimately on the fact that epimorphisms split in $\tilde{\mathbf{B}}$.

As Barr points out [1], the argument just outlined works equally well for sheaves of R -modules where R is a sheaf of rings. Perhaps a few words about the proof are in order. Since $QS(R \times R)$ is naturally isomorphic to $QSR \times QSR$, it follows that the adjunction $\eta R: R \rightarrow QSR$ is a ring homomorphism. Thus if N is a sheaf of R -modules then ηR makes QSN into a sheaf of R -modules in such a way that $\eta N: N \rightarrow QSN$ is an R -homomorphism. By induction, $(QS)^i N$ is a sheaf of R -modules and each $(QS)^i \eta (QS)^{n-i} N$ is an R -homomorphism. Letting G' be the "free sheaf of R -modules" cotriple, we can proceed as in the last paragraph by letting $C^{i,j} = \text{Hom}_R(G'^{i+1}M, (QS)^{j+1}N)$. Then the n th homology group of $C^{*,*}$ is in one-to-one correspondence with $\text{Ext}_R^n(M, N)$.

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