JACOBI POLYNOMIAL SUMS AND UNIVALENT CESÀRO MEANS

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ABSTRACT. We prove that Cesàro means of certain univalent power series are univalent. We establish a connection between these means and a conjecture of R. Askey about a type of Jacobi polynomial sum.

1. Let S be the class of functions

$$(1.1) \quad f(z) = z + a_2 z^2 + \cdots$$

that are analytic and univalent in the disc $E = \{z : |z| < 1\}$. We will consider the subclass $U \subset S$ that contains the functions with the property such that each partial sum

$$(1.2) \quad S_n(z; f) = z + a_2 z^2 + \cdots + a_n z^n$$

is also univalent in E . In general if $f \in S$ then $S_n(z; f)$ need not be univalent in E . As a matter of fact, G. Szegő [7] proved that the most that can be said in this regard is that if $f \in S$ then $S_n(z; f)$ is univalent in the disc $|z| < 1/4$. The functions in U form a rather small subclass of S , but by using a result of J. W. Alexander [1] it is easy to construct members of U . Alexander proved that $0 < a_{n+1} \leq a_n \leq 1$ implies that

$$(1.3) \quad f(z) = z + (a_2/2)z^2 + (a_3/3)z^3 + \cdots$$

belongs to U . In particular the function

$$(1.4) \quad f(z) = z + z^2/2 + z^3/3 + \cdots$$

belongs to U .

Most results in complex function theory that involve summability are representation theorems. There is relatively little known about the effect of summation methods on the geometric properties of complex functions. One result

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of this type is due to M. S. Robertson [6] who proved that if $f \in U$ then the $(C, 1)$ means of (1.1) are all univalent in E . Robertson uses a result on polynomial zeroes due to Obrechhoff to prove his theorem. Here we will give a simple self-contained proof that if $f \in U$ and k is a natural number then all the (C, k) means of (1.1) are univalent in E . Our proof, like Robertson's, uses the following criterion due to Dieudonné: The polynomial

$$(1.5) \quad P_n(z) = z + a_2 z^2 + \cdots + a_n z^n$$

is univalent in E if and only if the polynomial

$$(1.6) \quad Q_n(z; \theta) = 1 + \sum_{k=2}^n a_k \frac{\sin k\theta}{\sin \theta} z^{k-1}$$

has no zeroes for $z \in E$ and $0 < \theta < \pi$. We will refer to $Q_n(z; \theta)$ as the Dieudonné polynomial associated with $P_n(z)$. It is a simple consequence of this criterion that if $f \in U$ then $|a_n| \leq 1/n$. Thus the function (1.4) is extremal in U with respect to coefficient size. We conjecture that if $f \in U$ then the (C, α) means of (1.1) are univalent in E for all $\alpha \geq 0$, and we support the conjecture by proving it for the extremal function (1.4). This special case leads to a theorem involving Jacobi polynomials that is closely related to a conjecture of R. Askey.

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2. The functions in S are normalized so that $f'(0) = 1$. In order to preserve this normalization for the (C, α) means $P_n(z; \alpha)$ of (1.1) we define them by the identity

$$(2.1) \quad (1-w)^{-\alpha-1} \sum_{n=0}^{\infty} a_{n+1} z^{n+1} w^n = \sum_{n=0}^{\infty} \binom{n+\alpha}{n} P_{n+1}(z; \alpha) w^n.$$

Then we have

$$(2.2) \quad P_n(z; \alpha) = \binom{n+\alpha-1}{n-1}^{-1} \sum_{j=1}^n \binom{n-j+\alpha}{n-j} a_j z^j.$$

Our first theorem uses the following Lemma. This Lemma has been used in several places but we include a statement and proof for the sake of completeness.

Lemma. Let $P_n(z)$ be a polynomial of degree n and $P_n(z) \neq 0$, $z \in E$.

Let μ be a constant and set

$$Q_n(z) = P_n(z) - (z/\mu)P'_n(z).$$

Then $Q_n(z)$ has no zeroes in E when $\operatorname{Re} \mu > n/2$.

Proof. Let $\alpha_1, \dots, \alpha_n$ be the zeroes of $P_n(z)$ so that $|\alpha_j| \geq 1$. If $Q_n(z) = 0$ then $zP'_n/P_n = \mu$ and this implies

$$(2.3) \quad \sum_{j=1}^n \frac{z}{z - \alpha_j} = \mu.$$

But if $z \in E$ then $|\alpha_j| \geq 1$ implies that $\operatorname{Re}(z/(z - \alpha_j)) \leq 1/2$. Hence we get for $z \in E$ by (2.3) that $\operatorname{Re} zP'_n/P_n \leq n/2$. Since $\operatorname{Re} \mu > n/2$, $Q_n(z)$ cannot have zeroes in E .

Theorem 1. If $f \in U$ and k is a natural number, then the (C, k) means of (1.1) are univalent in E .

Proof. The proof is by induction on k . The case $k = 0$ is true because $f \in U$. We will only indicate the details. First observe that

$$(2.4) \quad \binom{n-j+k+1}{n-j} \bigg/ \binom{n+k}{n-1} = \frac{n-j+k+1}{n+k} \binom{n-j+k}{n-j} \bigg/ \binom{n+k-1}{n-1}.$$

Now let $Q_n(z; \theta, k)$ be the Dieudonné polynomial of $P_n(z; k)$. Then by (2.4) we have

$$\begin{aligned} Q_n(z; \theta, k+1) &= 1 + \binom{n+k}{n-1}^{-1} \sum_{j=2}^n \binom{n-j+k+1}{n-j} a_j \frac{\sin j\theta}{\sin \theta} z^{j-1} \\ &= 1 + \binom{n+k-1}{n-1}^{-1} \sum_{j=2}^n \frac{n-j+k+1}{n+k} \binom{n-j+k}{n-j} a_j z^{j-1}. \end{aligned}$$

Doing a little rewriting we get

$$(2.5) \quad Q_n(z; \theta, k+1) = Q_n(z; \theta, k) - (z/(n+k)) Q'_n(z; \theta, k).$$

Now we need only apply the Lemma to (2.5) repeatedly for $k = 0, 1, 2, \dots$, and this proves the theorem.

The proof of Theorem 1 does not extend to give the result for all $k \geq 0$. But it seems reasonable to suspect that Theorem 1 is true for all $k \geq 0$. For the function (1.4), which as we mentioned is extremal with respect to coefficient size, we have

Theorem 2. The (C, α) means of (1.4) are univalent in E for all $\alpha \geq 0$.

Proof. The (C, α) mean of (1.4) is given by

$$(2.6) \quad P_n(z; \alpha) = \binom{n+\alpha-1}{n-1}^{-1} \sum_{j=1}^n \binom{n-j+\alpha}{n-j} \frac{1}{j} z^j.$$

We will prove the univalence of (2.6) by applying Alexander's theorem as quoted in the introduction. Writing

$$b_j = \binom{n-j+\alpha}{n-j} \bigg/ \binom{n+\alpha-1}{n-1},$$

we need only show that $0 < b_{j+1} \leq b_j \leq 1$, $j = 1, \dots, n-1$. A short computation shows that $b_{j+1} \leq b_j$ is equivalent to $1 \leq (n-j+\alpha)/(n-j)$ which clearly holds for $\alpha \geq 0$. Since $b_1 = 1$ the result follows.

3. Let $P_k^{(\alpha, \beta)}(x)$ be the Jacobi polynomial of degree k with parameters α and β . R. Askey has made the following conjecture [2]:

$$(3.1) \quad \sum_{k=0}^n \frac{P_k^{(\alpha, \beta)}(\cos \theta)}{P_k^{(\beta, \beta)}(1)} z^k \neq 0 \quad \text{if } \beta > 0, \quad 0 \leq \theta \leq \pi, \quad z \in E.$$

Part of the reason for Askey's conjecture seems to have been G. Szegő's result [8] that (3.1) holds when $\beta = 0$. More specifically, when $\beta = 0$ the Jacobi polynomials reduce to the Legendre polynomials $P_k(x)$ and Szegő's theorem states that

$$\sum_{k=0}^n P_k(\cos \theta) z^k \neq 0 \quad \text{if } 0 \leq \theta \leq \pi, \quad z \in E.$$

When $\beta = 1/2$ the polynomial in (3.1) becomes

$$(3.2) \quad \sum_{k=0}^n \frac{\sin(k+1)\theta}{(k+1) \sin \theta} z^k$$

and (3.2) is the Dieudonné polynomial of the univalent polynomial

$$(3.3) \quad z + z^2/2 + \dots + z^{n+1}/(n+1).$$

Thus (3.2) has no zeroes for $z \in E$, $0 \leq \theta \leq \pi$, so that Askey's conjecture is true for $\beta = 1/2$ as well as for $\beta = 0$. On the other hand, we know from Theorem 2 that the Dieudonné polynomials of the (C, α) means of (1.4) do not vanish in E for $\alpha \geq 0$. Consequently we have the following result.

Theorem 3. If $\alpha \geq 0$, $z \in E$, and $0 \leq \theta \leq \pi$, then

$$(3.4) \quad \sum_{k=0}^n \binom{n-k+\alpha}{n-k} \frac{\sin(k+1)\theta}{(k+1)\sin\theta} z^k \neq 0.$$

Theorem 3 suggests a new conjecture that includes that of Askey. The conjecture is that if $\alpha \geq 0$, $\beta \geq 0$, $z \in E$, and $0 \leq \theta \leq \pi$, then

$$(3.5) \quad \sum_{k=0}^n \binom{n-k+\alpha}{n-k} \frac{P_k^{(\beta, \beta)}(\cos \theta)}{P_k^{(\beta, \beta)}(1)} z^k \neq 0.$$

The next result verifies (3.5) for some values of α and β .

Theorem 4. *If $\alpha \geq 1$, $\beta > \frac{1}{2}$, $z \in E$, and $0 \leq \theta \leq \pi$ then (3.5) holds.*

Proof. In [3] it is proved that if $\beta > \gamma \geq -\frac{1}{2}$ then

$$(3.6) \quad \frac{P_n^{(\beta, \beta)}(\cos \theta)}{P_n^{(\beta, \beta)}(1)} = \int_0^\pi \frac{P_n^{(\gamma, \gamma)}(\cos \theta)}{P_n^{(\gamma, \gamma)}(1)} d\mu(\theta)$$

where $d\mu(\theta)$ is a positive measure. Then denoting the polynomials in (3.5) by $P_n(z; \alpha, \beta, \theta)$ and using (3.6) we have for $\beta > \frac{1}{2}$

$$(3.7) \quad P_n(z; \alpha, \beta, \theta) = \int_0^\pi \sum_{k=0}^n \binom{n-k+\alpha}{n-k} \frac{P_k^{(\frac{1}{2}, \frac{1}{2})}(\cos \theta)}{P_k^{(\frac{1}{2}, \frac{1}{2})}(1)} z^k d\mu(\theta).$$

We will show that $P_n(z; \alpha, \beta, \theta) \neq 0$ by showing that

$$(3.8) \quad \operatorname{Im} z P_n(z; \alpha, \beta, \theta) > 0, \quad z \in E, \quad \operatorname{Im} z > 0, \quad 0 < \theta < \pi.$$

By writing $z P_n(z; \alpha, \beta, \theta)$ as a Poisson integral we can conclude from (3.8) and the positivity of the Poisson kernel that $z P_n(z; \alpha, \beta, \theta) = 0$ in E only when $z = 0$. We proceed then to prove (3.8). Recalling that

$$\frac{P_k^{(\frac{1}{2}, \frac{1}{2})}(\cos \theta)}{P_k^{(\frac{1}{2}, \frac{1}{2})}(1)} = \frac{\sin(k+1)\theta}{(k+1)\sin\theta},$$

we have

$$(3.9) \quad z P_n(z; \alpha, \beta, \theta) = \int_0^\pi \sum_{k=0}^n \binom{n-k+\alpha}{n-k} \frac{\sin(k+1)\theta}{(k+1)\sin\theta} z^{k+1} d\mu(\theta).$$

Writing $z = re^{i\phi}$ we want to show that if $0 < r < 1$, $0 < \phi < \pi$, $0 < \theta < \pi$ then (after a minor change in index)

$$(3.10) \quad \sum_{k=1}^n \binom{n-k+\alpha}{n-k} \frac{\sin k\theta}{k \sin \theta} \sin k\phi r^k > 0.$$

By applying a useful lemma of Fejér [4] we have (3.10) if we can show that

$$(3.11) \quad \sum_{k=1}^n \binom{n-k+\alpha}{n-k} \frac{\sin k\theta}{\sin \theta} r^k > 0, \quad 0 < \theta < \pi, \quad 0 < r < 1.$$

But (3.11) holds for $0 < r < 1$ if [8]

$$(3.12) \quad \sum_{k=1}^n \binom{n-k+\alpha}{n-k} \frac{\sin k\theta}{\sin \theta} > 0, \quad 0 < \theta < \pi.$$

Now we need to refer to an old result of Lukacs [5]. He proved that

$$(3.13) \quad \sum_{k=1}^n (n-k+1) \sin k\theta > 0, \quad 0 < \theta < \pi.$$

Inequality (3.13) implies inequality (3.12) for $\alpha \geq 1$ and working back we have proved (3.8) and hence (3.5).

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