MARKUŠEVIČ BASES IN SOME DUAL SPACES

K. JOHN AND V. ZIZLER

ABSTRACT. If X^* is locally uniformly rotund, then X^* has a Markuševič basis.

It was proved by Tacon [8] that if the norm of a Banach space X is Fréchet differentiable, then X^* admits a projectional resolution $\{P_\alpha\}$ of the identity I . The same result was shown in [4] if X admits a continuously Fréchet differentiable function with bounded nonempty support. Here we prove that if X^* is locally uniformly rotund, then the projections $(P_{\alpha+1} - P_\alpha)$ have also nice ranges for transfinite induction, which enables us to derive the result mentioned in the Abstract.

The above mentioned results lead to the following questions: (a) Does X^* have a Markuševič basis if X has Fréchet differentiable norm? (b) Does there exist a locally uniformly rotund Banach space without Markuševič basis? (There is a strictly convex space without Markuševič basis by [2].) Also, it should be remarked that the result mentioned in the Abstract cannot be strengthened to give a Markuševič basis on X^* with coefficients from X (i.e. shrinking Markuševič basis on X), since there is a Banach space X which is not weakly compactly generated and X^* is locally uniformly rotund [6].

A Banach space X is locally uniformly rotund (LUR) if whenever x_n , $x \in X$, $|x_n| = |x| = 1$, $\lim |x_n + x| = 2$, then $\lim |x_n - x| = 0$. If M is a subset of a Banach space X , then $\text{cl } M$ means the norm closure of M in X and $\overline{\text{sp}} M$ is the norm closed linear hull of M . $\text{dens } X$ is the smallest cardinality of a dense subset of X . Symbol $\cong$ denotes the linear isometry. A Markuševič basis in a Banach space X is a family $\{x_\alpha\} \subset X$, $\{f_\alpha\} \subset X^*$, $\alpha \in \Gamma$ such that $f_\alpha(x_\beta) = \delta_{\alpha\beta}$, $\overline{\text{sp}} \bigcup_\alpha x_\alpha = X$ and $\{f_\alpha\}$ total on X (i.e. $\bigcap_\alpha f_\alpha^{-1}(0) = 0$). Unless stated otherwise, in X^* we take the dual sup-norm.

Theorem. If X^* is LUR, then there is a transfinite sequence of continuous linear projections $Q_\alpha: X^* \rightarrow X^*$, $0 \leq \alpha \leq \nu$, with

- (1) $Q_0 = 0$, $Q_\nu = I$;
- (2) $Q_\alpha Q_\beta = Q_\beta Q_\alpha = Q_\alpha$ if $\alpha < \beta$;

Received by the editors April 24, 1974.

AMS (MOS) subject classifications (1970). Primary 46B99.

Key words and phrases. Markuševič basis, locally uniformly rotund spaces.

Copyright © 1975, American Mathematical Society

- (3) for any $x^* \in X^*$ and $0 \leq \alpha \leq \gamma$ we have $Q_\alpha x^* \in \overline{\text{sp}} \bigcup_{\beta < \alpha} Q_{\beta+1} x^*$;
 (4) for any $x^* \in X^*$ and $\epsilon > 0$, the set

$$\{\alpha, |(Q_{\alpha+1} - Q_\alpha)x^*| \geq \epsilon(|Q_{\alpha+1}| + |Q_\alpha|)\}$$

is finite;

$$(5) \text{ dens}(Q_{\alpha+1} - Q_\alpha)X^* \leq \aleph_0.$$

Corollary. If X^* is LUR, then X^* has a Markuševič basis.

Proof of Corollary. If for $0 \leq \alpha < \gamma$, $x_{\alpha,n} \in (Q_{\alpha+1} - Q_\alpha)X^*$, $f_{\alpha,n} \in ((Q_{\alpha+1} - Q_\alpha)X^*)^*$, $n = 1, 2, \dots$, is a Markuševič basis of $(Q_{\alpha+1} - Q_\alpha)X^*$ (cf., e.g., [5]), then it is easy to see that $\bigcup_{\alpha,n} x_{\alpha,n}$, $\bigcup_{\alpha,n} (Q_{\alpha+1} - Q_\alpha)^* f_{\alpha,n}$, $0 \leq \alpha < \gamma$, $n = 1, 2, \dots$, is a Markuševič basis of X^* (where $(Q_{\alpha+1} - Q_\alpha)^*$: $((Q_{\alpha+1} - Q_\alpha)X^*)^* \rightarrow X^{**}$).

For proving the Theorem we will need the following two lemmas. The first one is a modification of Tacon's Theorem 2 [8]. We use Lovaglia's result that the norm of a Banach space is Fréchet differentiable if X^* is LUR [7].

Lemma 1. Let X^* be LUR and μ be the first ordinal of cardinality $\text{dens } X (= \text{dens } X^*)$. Then for any $0 \leq \alpha \leq \mu$ there is a subspace $X_\alpha \subset X$ together with a linear operator $T_\alpha: X_\alpha^* \rightarrow X^*$ such that

- (1) $X_\alpha \subset X_\beta$ if $\alpha < \beta$, $X_0 = 0$, $X_\mu = X$, $|T_\alpha| = 1$ for $\alpha > 0$, $T_0 = 0$, $T_\mu = I$, $T_\alpha X^* \subset T_\beta X^*$ if $\alpha < \beta$, $\text{dens } X_\alpha \leq \bar{\alpha}$ for infinite α ;
 (2) $X_\alpha = \text{cl} \bigcup_{\beta < \alpha} X_{\beta+1}$;
 (3) $T_\alpha^* x = x$ for $x \in X_\alpha$, $\alpha > 0$; and

if we denote by $P_\alpha = T_\alpha R_\alpha: X^* \rightarrow X^*$, for R_α the restriction of X^* to X_α^* and $\alpha > 0$, $P_0 = 0$, then

- (4) P_α is a projection;
 (5) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\alpha < \beta$, $P_\alpha^* x = x$ for $x \in X_\alpha$, $\alpha > 0$;
 (6) for $x^* \in X^*$, $P_\alpha x^*$ is norm continuous on ordinals;
 (7) $P_\alpha X^* = \{f \in X^*, |f| = |R_\alpha f|\} \equiv A_\alpha$;
 (8) $(P_{\alpha+1} - P_\alpha)X^* = (A_{\alpha+1} \cap X_\alpha^\perp) \cong (X_{\alpha+1}/X_\alpha)^*$.

Proof. (1)–(3) was proved by Tacon [8], using the arguments of [1]. (5) follows from (1) and (3) [8]. We prove (7). If $f = T_\alpha g$, $g \in X_\alpha^*$, $|g| = 1$, then $f(x) = (T_\alpha g)(x) = g(x)$ for $x \in X_\alpha$ by (3). So, $|R_\alpha f| = 1$ and from $|T_\alpha| = 1$, also $|f| = |T_\alpha g| = 1$. If $|f| = |R_\alpha f| = 1$, then using (3), we have $2 \geq |f + P_\alpha f| \geq 2|R_\alpha f| = 2$, so, by LUR of X^* , $f = P_\alpha f$. Thus (7) holds. To prove

(4) it suffices to show that if $|f| = |R_\alpha f| = 1$, then $f = P_\alpha f$. If $|f| = |R_\alpha f| = 1$, then, using (1) and (3), $2 \geq |f + P_\alpha f| \geq 2|R_\alpha f| = 2$, so, by LUR of X^* , $f = P_\alpha f$.

We now prove that $P_\alpha X^* = \text{cl } \bigcup_{\beta < \alpha} P_\beta X^*$ if α is limiting ordinal. For if $f \in P_\alpha X^*$, $|f| = 1$, then by (7), $|R_\alpha f| = 1$ and easily from (2), $\lim_{\beta \rightarrow \alpha} |R_\beta f| = 1$. Let $\text{ER}_\beta f$ denote a norm preserving Hahn-Banach extension of $R_\beta f$ onto X . Then $\text{ER}_\beta f \in P_\beta X^*$ by (7). Moreover, $|\text{ER}_\beta f + f| \geq 2|R_\beta f|$, so $\lim_{\beta \rightarrow \alpha} |\text{ER}_\beta f + f| = 2$ and by LUR of X^* , $\lim_{\beta \rightarrow \alpha} |\text{ER}_\beta f - f| = 0$, and thus $P_\alpha X = \text{cl } \bigcup_{\beta < \alpha} P_\beta X^*$. From this and from $|P_\alpha| \leq 1$, (6) follows (cf. e.g., [3, Lemma 2]). To prove (8), let $f \in (P_{\alpha+1} - P_\alpha)X^*$. Then $f \in P_{\alpha+1}X^*$ and thus by (7), $f \in A_{\alpha+1}$. Furthermore, $P_\alpha f = 0$ and thus if $x \in X_\alpha$, then by (3), $f(x) = f(P_\alpha^* x) = (P_\alpha f)(x) = 0$. If $f \in A_{\alpha+1} \cap X_\alpha^\perp$, then $f \in P_{\alpha+1}X^*$ by (7). Moreover $P_\alpha f = T_\alpha R_\alpha f = T_\alpha 0 = 0$, so $(P_{\alpha+1} - P_\alpha)f = P_{\alpha+1}f = f$. The mapping $R_{\alpha+1}$ then serves to show the isometry in (8).

Lemma 2. Let X^* be LUR. Then if $Z \subset Y \subset X$ are subspaces of X , then $(Y/Z)^*$ is LUR.

Proof. First, $Y^* = (X^*/Y^\perp)$. We show $X^*/Y^\perp$ is LUR. For it, let $\hat{x}_n \in X/Y^\perp$, $|\hat{x}| = |\hat{x}_n| = 1$, $\lim |\hat{x} + \hat{x}_n| = 2$. Let $x \in \hat{x}$, $x_n \in \hat{x}_n$, be so that $|x| = |x_n| = 1$. Such points exist since $Y^\perp$ is w^* closed. Then $2 \geq |x + x_n| \geq |\hat{x} + \hat{x}_n|$. Thus $\lim |x + x_n| = 2$ and by LUR of X^* , $\lim |x - x_n| \geq \lim |\hat{x} - \hat{x}_n| = 0$. Furthermore $(Y/Z)^* \cong Z^\perp$ taken in Y^* , so it is LUR.

Proof of Theorem. By transfinite induction on $\text{dens } X^*$. If $\text{dens } X^* = \aleph_0$, then we can take $Q_0 = 0$, $Q_1 = I$. Assume $\text{dens } X^* = \aleph$ and that the Theorem holds for any Y^* with $\text{dens } Y^* < \aleph$. There are projections $\{P_\alpha\}$, $\alpha < \mu$, from Lemma 1. By transfinite induction hypothesis and Lemmas 1 ((8), (1)) and 2, there is for any $0 \leq \alpha < \mu$ a transfinite sequence of continuous linear projections

$$S_{\alpha, \beta} : (P_{\alpha+1} - P_\alpha)X^* \rightarrow (P_{\alpha+1} - P_\alpha)X^*, \quad \beta \in \Gamma_\alpha,$$

with properties (1)–(5) of the Theorem. If $\{(\alpha, \beta)\}$ is a family of pairs (α, β) , $\alpha < \mu$, $\beta \in \Gamma_\alpha$, and $(\alpha, 0)$ if $\alpha = \mu$, ordered lexicographically, then the projections $Q_{(\alpha, \beta)} : X^* \rightarrow X^*$ defined by

$$Q_{(\alpha, \beta)} = S_{\alpha, \beta}(P_{\alpha+1} - P_\alpha) + P_\alpha \quad \text{for } \alpha < \mu$$

and

$$Q_{(\mu, 0)} = P_\mu = I \quad (\text{see [9], [10]}),$$

satisfy (1)–(5) of the Theorem [9, p. 177], [10, p. 128].

REFERENCES

1. D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. (2) 88 (1968), 35–46. MR 37 #4562.
2. F. K. Dashiell and J. Lindenstrauss, *Some examples concerning strictly convex norms on $C(K)$ spaces*, Israel J. Math. 16 (1973), 329–342.
3. K. John and V. Zizler, *Smoothness and its equivalents in the class of weakly compactly generated Banach spaces*, J. Functional Analysis 15 (1974), 1–11.
4. ———, *Duals of Banach spaces which admit nontrivial smooth functions*, Bull. Austral. Math. Soc. 11 (1974), 161–166.
5. W. B. Johnson, *Markušević bases and duality theory*, Trans. Amer. Math. Soc. 149 (1970), 171–177. MR 41 #5927.
6. W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces* (to appear).
7. A. R. Lovaglia, *Locally uniformly convex Banach spaces*, Trans. Amer. Math. Soc. 78 (1955), 225–238. MR 16, 596.
8. D. G. Tacon, *The conjugate of a smooth Banach space*, Bull. Austral. Math. Soc. 2 (1970), 415–425. MR 41 #5942.
9. S. Trojanski, *On locally uniformly convex and differentiable norms in certain nonseparable Banach spaces*, Studia Math. 37 (1970), 169–176.
10. ———, *On equivalent norms and minimal systems in nonseparable Banach spaces*, Studia Math. 43 (1972), 125–138.

MATHEMATICAL INSTITUTE, CZECHOSLOVAK ACADEMY OF SCIENCES, PRAGUE,
CZECHOSLOVAKIA

DEPARTMENT OF MATHEMATICS, CHARLES UNIVERSITY, PRAGUE, CZECHOSLO-
VAKIA