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$L_{p,q}$ MODULARS*

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ABSTRACT. For $1 < p \leq q$, a convex modular m on a linear space S is called an $L_{p,q}$ modular if $\text{Min}_{r=p,q} \xi^r m(x) \leq m(\xi x) \leq \text{Max}_{r=p,q} \xi^r m(x)$ for $\xi > 0$ and $x \in S$. We generalize the Minkowski inequality and the Hölder inequality for $L_{p,q}$ modulars.

For $1 < p \leq q$, a convex modular m on a linear space S is called an $L_{p,q}$ modular if

$$\xi^p m(x) \leq m(\xi x) \leq \xi^q m(x) \quad \text{for } \xi \geq 1 \text{ and } x \in S,$$

and

$$\xi^p m(x) \geq m(\xi x) \geq \xi^q m(x) \quad \text{for } 0 < \xi \leq 1 \text{ and } x \in S.$$

An $L_{p,q}$ modular is p -homogeneous if $p = q$, as defined in [3]. We generalized the Minkowski inequality and the Hölder inequality for p -homogeneous modu-

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lars in [3] by the theory of modular spaces in [1] and [2]. In this paper, we will generalize them for $L_{p,q}$ modulars.

For the second modular norm N_2 of m , we have

$$N_2(x) = \inf_{m(\xi x) \leq 1, \xi > 0} \frac{1}{\xi} \quad \text{for } x \in S,$$

and we have $N_2(x) \leq 1$ if and only if $m(x) \leq 1$, as proved in [1] and [2]. If $m(x) \leq 1$, then it is clear that $N_2(x) = \inf_{m(\xi x) \leq 1, \xi \geq 1} (1/\xi)$. Since $\xi^p m(x) \leq m(\xi x) \leq \xi^q m(x)$ for $\xi \geq 1$, we obtain

$$m(x)^{1/p} = \inf_{\xi^p m(x) \leq 1, \xi \geq 1} \frac{1}{\xi} \leq N_2(x) \leq \inf_{\xi^q m(x) \leq 1, \xi \geq 1} \frac{1}{\xi} = m(x)^{1/q}, \quad \text{if } m(x) \leq 1.$$

That is, $m(x)^{1/p} \leq N_2(x) \leq m(x)^{1/q}$ if $m(x) \leq 1$. Likewise, we obtain $m(x)^{1/q} \leq N_2(x) \leq m(x)^{1/p}$ if $m(x) \geq 1$. Therefore we have

$$\text{Min}_{r=p,q} m(x)^{1/r} \leq N_2(x) \leq \text{Max}_{r=p,q} m(x)^{1/r} \quad \text{for } x \in S.$$

Since $N_2(x+y) \leq N_2(x) + N_2(y)$, we obtain a generalization of the Minkowski inequality

$$(1) \quad \text{Min}_{r=p,q} m(x+y)^{1/r} \leq \text{Max}_{r=p,q} m(x)^{1/r} + \text{Max}_{r=p,q} m(y)^{1/r}.$$

For the first modular norm N_1 of m , we have

$$N_1(x) = \inf_{\xi > 0} \frac{1}{\xi} (1 + m(\xi x)) \quad \text{for } x \in S,$$

as proved in [1] and [2]. If $N_1(x) = \inf_{1 \geq \xi > 0} (1/\xi) (1 + m(\xi x))$, then

$$\begin{aligned} q^{1/q} q'^{1/q'} m(x)^{1/q} &= \inf_{\xi > 0} \frac{1}{\xi} (1 + \xi^q m(x)) \leq N_1(x) \\ &\leq \inf_{\xi > 0} \frac{1}{\xi} (1 + \xi^p m(x)) = p^{1/p} p'^{1/p'} m(x)^{1/p} \end{aligned}$$

for $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. If $N_1(x) = \inf_{\xi \geq 1} (1/\xi) (1 + m(\xi x))$, then

$$p^{1/p} p'^{1/p'} m(x)^{1/p} \leq N_1(x) \leq q^{1/q} q'^{1/q'} m(x)^{1/q}.$$

Therefore

$$\text{Min}_{r=p,q} r^{1/r} r'^{1/r'} m(x)^{1/r} \leq N_1(x) \leq \text{Max}_{r=p,q} r^{1/r} r'^{1/r'} m(x)^{1/r},$$

for $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, and $x \in S$.

Let $\bar{S}$ be the conjugate space of S and let $\bar{m}$ be the conjugate modular of an $L_{p,q}$ modular m : $\bar{m}(\phi) = \sup_{x \in S} \{\phi(x) - m(x)\}$ for every $\phi \in \bar{S}$. By definition,

$$\bar{m}(\xi\phi) = \sup_{x \in S} \{\phi(\xi x) - m(x)\} = \sup_{x \in S} \{\phi(x) - m(x/\xi)\}$$

for every $\xi \neq 0$. In case $\xi \geq 1$, $m(x/\xi) \geq \xi^{\alpha p} m(x/\xi^{1+\alpha})$ if $\alpha = 1/(p-1)$. Then,

$$\bar{m}(\xi\phi) \leq \sup_{x \in S} \{\phi(x) - \xi^{\alpha p} m(x/\xi^{1+\alpha})\} = \xi^{\alpha p} \sup_{x \in S} \{\phi(x/\xi^{\alpha p}) - m(x/\xi^{1+\alpha})\}.$$

Let $1/p + 1/p' = 1$. Then $p/(p-1) = 1 + 1/(p-1) = p'$ and by replacing αp and $\alpha + 1$ in the above inequality, with p' ,

$$\bar{m}(\xi\phi) \leq \xi^{p'} \sup_{x \in S} \{\phi(x) - m(x)\} = \xi^{p'} \bar{m}(\phi).$$

In a similar manner, we can prove that $\bar{m}(\xi\phi) \geq \xi^{q'} \bar{m}(\phi)$ for q' such that $1/q + 1/q' = 1$. Therefore, if m is an $L_{p,q}$ modular, and $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$ then $\xi^{q'} \bar{m}(\phi) \leq \bar{m}(\xi\phi) \leq \xi^{p'} \bar{m}(\phi)$ for every $\xi \geq 1$ and every $\phi \in \bar{S}$. A similar computation also leads us to the following inequality, in case $0 < \xi \leq 1$:

$$\xi^{p'} \bar{m}(\phi) \leq \bar{m}(\xi\phi) \leq \xi^{q'} \bar{m}(\phi) \quad \text{for every } \phi \in \bar{S}.$$

Hence the conjugate modular $\bar{m}$ of an $L_{p,q}$ modular m , is an $L_{q',p'}$ modular.

For the first and second modular norms $\bar{N}_1$ and $\bar{N}_2$ of $\bar{m}$, we have

$$|\phi(x)| \leq \bar{N}_1(\phi) N_2(x) \quad \text{and} \quad |\phi(x)| \leq \bar{N}_2(\phi) N_1(x)$$

for $\phi \in \bar{S}$ and $x \in S$. Therefore we can generalize the Hölder inequality as

$$(2) \quad |\phi(x)| \leq \text{Max}_{r=p,q} r^{1/r, 1/r'} \bar{m}(\phi)^{1/r'} \text{Max}_{r=p,q} m(x)^{1/r},$$

and

$$(3) \quad |\phi(x)| \leq \text{Max}_{r=p,q} \bar{m}(\phi)^{1/r'} \text{Max}_{r=p,q} r^{1/r, 1/r'} m(x)^{1/r}.$$

As an example, we consider the linear space S_0 of all continuous functions on the closed interval $[0, 1]$. For $\rho \in S_0$ with $1 < p \leq \rho(t) \leq q$ for $t \in [0, 1]$, setting

$$m(\phi) = \int_0^1 |\phi(t)|^{\rho(t)} dt \quad \text{for } \phi \in S_0,$$

we obtain an $L_{p,q}$ modular m on S_0 . By (1) we have a generalization of the Minkowski inequality

$$\begin{aligned} & \text{Min}_{r=p,q} \left(\int_0^1 |\phi(t) + \psi(t)|^{\rho(t)} dt \right)^{1/r} \\ & \leq \text{Max}_{r=p,q} \left(\int_0^1 |\phi(t)|^{\rho(t)} dt \right)^{1/r} + \text{Max}_{r=p,q} \left(\int_0^1 |\psi(t)|^{\rho(t)} dt \right)^{1/r} \end{aligned}$$

if $1 < p \leq \rho(t) \leq q$ for $t \in [0, 1]$.

We also set

$$m_0(\phi) = \int_0^1 \frac{1}{\rho(t)} |\phi(t)|^{\rho(t)} dt \quad \text{for } \phi \in S_0.$$

Then m_0 is an $L_{p,q}$ modular on S_0 . For the conjugate space $\bar{S}_0$ of S_0 for m_0 , we can consider $S_0 \subset \bar{S}_0$ by setting $\psi(\phi) = \int_0^1 \psi(t)\phi(t)dt$ for $\phi, \psi \in S_0$, and the conjugate modular $\bar{m}_0$ of m_0 is given by

$$\bar{m}_0(\psi) = \int_0^1 \frac{1}{\rho'(t)} |\psi(t)|^{\rho'(t)} dt \quad \text{for } \psi \in S_0$$

if $1/\rho(t) + 1/\rho'(t) = 1$ for $t \in [0, 1]$, as proved in [1]. Thus, by (2) and (3), we can generalize the Hölder inequality as

$$\begin{aligned} & \int_0^1 |\psi(t)\phi(t)| dt \\ & \leq \max_{r=p,q} r^{1/r} r'^{1/r'} \left(\int_0^1 \frac{1}{\rho'(t)} |\psi(t)|^{\rho'(t)} dt \right)^{1/r'} \max_{r=p,q} \left(\int_0^1 \frac{1}{\rho(t)} |\phi(t)|^{\rho(t)} dt \right)^{1/r} \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 |\psi(t)\phi(t)| dt \\ & \leq \max_{r=p,q} \left(\int_0^1 \frac{1}{\rho'(t)} |\psi(t)|^{\rho'(t)} dt \right)^{1/r'} \max_{r=p,q} r^{1/r} r'^{1/r'} \left(\int_0^1 \frac{1}{\rho(t)} |\phi(t)|^{\rho(t)} dt \right)^{1/r} \end{aligned}$$

if $1 < p \leq \rho(t) \leq q$ and $1/\rho(t) + 1/\rho'(t) = 1$ for $t \in [0, 1]$, $1/p + 1/p' = 1$, and $1/q + 1/q' = 1$.

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