THE SPECTRUM OF SOME SPECIAL ELEMENTS IN THE FREE BANACH ALGEBRA

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ABSTRACT. Let F be the free semigroup on two generators, and let e denote the empty word in F . It is shown that if $f \in l'(F)$ has the property that its support freely generates a subsemigroup of F , then the spectrum of f in $l'(F)$ is equal to $\{\lambda: |\lambda - f(e)| \leq \sum_{s \neq e} |f(s)|\}$.

Introduction. Let F denote the free semigroup on two generators x and y with an identity e adjoined to it (e is usually referred to as the empty word). Let Γ be the algebra of all complex valued functions on F with convolution multiplication, i. e.

$$fg(s) = \sum_{rt=s} f(r)g(t)$$

for all $f, g \in \Gamma$ and $s \in F$.

It is easy to see that the element of Γ whose value is 1 at e and 0 elsewhere serves as an identity for Γ , and with an easy computation one can check that an element $f \in \Gamma$ is invertible if and only if $f(e) \neq 0$.

Let $A = l'(F)$ be the subalgebra of Γ consisting of all $f \in \Gamma$ with the property that $\sum_{s \in F} |f(s)| < \infty$. Then under the usual l' norm, A is a Banach algebra with identity.

Considering the generator x as an element of A whose value is 1 at x and 0 elsewhere, it can be easily shown that $\text{Sp } x = \{\lambda: |\lambda| \leq 1\}$, where $\text{Sp } x$ denotes the spectrum of x in A . In this paper we generalize this fact to a special class of elements of A that includes the generators x and y .

Given $f \in \Gamma$ we let S_f denote the support of f in F , i.e. $S_f = \{s \in F: f(s) \neq 0\}$, and we let G_f be the subsemigroup of F generated by S_f . We say f is *free* if G_f is freely generated by S_f . Free elements in A are easily constructed, in fact every homogeneous polynomial is free [1, Chapter II], where an element $f \in \Gamma$ is a homogeneous polynomial if S_f consists of

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words of the same degree (the degree of an element $s \in F$ is the number of occurrences of x plus the number of occurrences of y in the expression for s).

Lemma 1. *Let $f \in \Gamma$ and suppose that $f(e) \neq 0$. Then f has an inverse, f^{-1} , in Γ .*

To prove this lemma one needs to check that there is a $g \in \Gamma$ such that $gf(e) = fg(e) = 1$ and $gf(s) = fg(s) = 0$ for $s \neq e$. This could be done by a straightforward computation which we omit [1, Chapter II].

Lemma 2. *Let $f \in \Gamma$ with $f(e) \neq 0$. Then $S_{f^{-1}} \subset G_f$.*

Proof. Let $k = \min\{d(s) : s \in S_f, s \neq e\}$, where $d(s)$ denotes the degree of s .

Suppose $r \notin G_f$ (so in particular $r \neq e$) and that $d(r) \leq k$. Then $0 = f^{-1}f(r) = f^{-1}f(r)f(e)$ and, hence, $f^{-1}f(r) = 0$ ($f^{-1}f$ is the identity which has value 0 except at e).

We now proceed by induction. Suppose that $f^{-1}f(r) = 0$ whenever $r \notin G_f$ and $d(r) \leq n$.

Let $d(r) = n + 1$ and $r \notin G_f$. If $f^{-1}f(r) \neq 0$, then since $f(r) = 0 = f^{-1}f(r)$ and $f^{-1}f(r)f(e) \neq 0$, there exists $s, t \in F$ such that $s \neq e \neq t$, $st = r$ and $f^{-1}f(s)f(t) \neq 0$. But then $d(s) < d(r) = n + 1$. Hence, by the induction hypothesis, $s \in G_f$, and thus $r = st \in G_f$, which is a contradiction.

Lemma 3. *Let $f \in A$ and let λ be a complex number. If $\lambda \in \text{Sp } f$ then $\sum_{s \neq e} |f(s)| \geq |\lambda - f(e)|$.*

Proof. Suppose that $\sum_{s \neq e} |f(s)| < |\lambda - f(e)|$. Then $\lambda \neq f(e)$. Let $g = [1/(\lambda - f(e))] \cdot (\lambda - f)$. Then $g(e) = 1$ and $\sum_{s \neq e} |g(s)| < 1$. Hence

$$\|1 - g\| = |1 - g(e)| + \sum_{s \neq e} |g(s)| = \sum_{s \neq e} |g(s)| < 1$$

(where $\|\cdot\|$ denotes the l^1 norm of A). Therefore g is invertible in A [2, 1.4.5], and thus $\lambda - f$ is invertible in A , i.e. $\lambda \notin \text{Sp } f$.

Lemma 4. *Let $f \in A$ be free. Suppose that S_f is finite and that $f(e) = 1$. Then f is invertible in A if and only if $\sum_{s \neq e} |f(s)| < 1$.*

Proof. Since $f(e) \neq 0$, f has an inverse, f^{-1} , by Lemma 1. By Lemma 2, we have $S_{f^{-1}} \subset G_f$.

Let $S_f = \{s_1, \dots, s_n\} \cup \{e\}$. Since f is free, every element in $S_{f^{-1}}$ could be written uniquely in the form $t_1 \cdots t_m$, where for each i , $t_i = s_j$ for some $j = 1, \dots, n$.

We have $1 = f^{-1}f(e) = f^{-1}(e)f(e) = f^{-1}(e)$.

Claim. If $s = t_1 \cdots t_m \neq e$, $t_i \in S_f$; then $f^{-1}(s) = (-1)^m \prod_{i=1}^m f(t_i)$.

We prove the claim by induction on m .

If $m = 1$, then we have

$$0 = f^{-1}f(t_1) = f^{-1}(t_1)f(e) + f^{-1}(e)f(t_1) = f^{-1}(t_1) + f(t_1)$$

(by the freeness of f , t_1 cannot be written as the product of other elements of S_f). Hence $f^{-1}(t_1) = -f(t_1)$.

Now suppose that $f^{-1}(t_1 \cdots t_k) = (-1)^k \prod_{i=1}^k f(t_i)$, and let $s = t_1 \cdots t_{k+1} \neq e$. Then

$$0 = f^{-1}f(s) = f^{-1}(s)f(e) + f^{-1}(t_1 \cdots t_k)f(t_{k+1})$$

(we used the uniqueness of the expression for s as a product of elements in S_f). Hence

$$\begin{aligned} f^{-1}(s) &= -f^{-1}(t_1 \cdots t_k)f(t_{k+1}) \\ &= - \left[(-1)^k \prod_{i=1}^k f(t_i) \right] f(t_{k+1}) = (-1)^{k+1} \prod_{i=1}^{k+1} f(t_i). \end{aligned}$$

This proves the claim.

Now given nonnegative integers $\alpha_1, \dots, \alpha_n$; there are

$$(\alpha_1 + \cdots + \alpha_n)! / \alpha_1! \cdots \alpha_n!$$

words in G_f which contains α_i factors of s_i for $i = 1, \dots, n$. All these words are distinct since f is free. Hence

$$\begin{aligned} \|f^{-1}\| &= \sum_{\alpha_1=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} \left[\frac{(\alpha_1 + \cdots + \alpha_n)!}{\alpha_1! \cdots \alpha_n!} \right] |f(s_1)^{\alpha_1} \cdots f(s_n)^{\alpha_n}| \\ &= \sum_{p=0}^{\infty} \left(\sum_{i=1}^n |f(s_i)| \right)^p = \sum_{p=0}^{\infty} \left(\sum_{s \neq e} |f(s)| \right)^p, \end{aligned}$$

and therefore $\|f^{-1}\| < \infty$ if and only if $\sum_{s \neq e} |f(s)| < 1$.

Lemma 5. Let $f \in A$ be free. Suppose that $f(e) = 1$. If f is invertible in A , then $\sum_{s \neq e} |f(s)| \leq 1$.

Proof. Since S_f generates a free semigroup, every subset of S_f generates a free semigroup. Hence f can be approximated by free elements with finite support.

Let δ be a positive number less than $1/\|f^{-1}\|$ and choose g free with finite support such that $g(e) = 1$ and $\|f - g\| < \delta$. Since $\delta < 1/\|f^{-1}\|$, g is invertible in A [2, 1.4.6], and thus by Lemma 4, $\sum_{s \neq e} |g(s)| < 1$. Hence

$$\sum_{s \neq e} |f(s)| = \|f\| - 1 \leq \|f - g\| + \|g\| - 1 = \|f - g\| + \sum_{s \neq e} |g(s)| < \delta + 1.$$

Theorem. Let $f \in A$ be free. Then $\text{Sp } f = \{\lambda: |\lambda - f(e)| \leq \sum_{s \neq e} |f(s)|\}$.

Proof. By Lemma 3 we have $\text{Sp } f \subset \{\lambda: |\lambda - f(e)| \leq \sum_{s \neq e} |f(s)|\}$.

Suppose $|\lambda - f(e)| < \sum_{s \neq e} |f(s)|$. If $\lambda = f(e)$, then clearly $\lambda \in \text{Sp } f$. Hence we may assume that $\lambda \neq f(e)$.

Let $g = 1/[\lambda - f(e)] \cdot (\lambda - f)$. Then $g(e) = 1$, g is free and $\sum_{s \neq e} |g(s)| = [1/|\lambda - f(e)|] \sum_{s \neq e} |f(s)| > 1$. Hence, by Lemma 5, g is not invertible in A , and thus $\lambda - f$ is not invertible in A , i.e. $\lambda \in \text{Sp } f$. Hence $\{\lambda: |\lambda - f(e)| < \sum_{s \neq e} |f(s)|\} \subset \text{Sp } f$. But since $\text{Sp } f$ is closed [2, 1.6.4] we get $\{\lambda: |\lambda - f(e)| \leq \sum_{s \neq e} |f(s)|\} \subset \text{Sp } f$, which completes the proof.

We now give an example of an element of A (necessarily nonfree) that does not satisfy the Theorem.

Let $f(e) = 1$, $f(x) = 2/3$, $f(x^2) = 4/9 = (2/3)^2$ and $f(s) = 0$ for all other $s \in F$. Then f has an inverse in Γ since $f(e) \neq 0$.

We have $1 = f^{-1}f(e) = f^{-1}(e)f(e) = f^{-1}(e)$, and

$$0 = f^{-1}f(x) = f^{-1}(e)f(x) + f^{-1}(x)f(e) = f(x) + f^{-1}(x).$$

Hence $f^{-1}(x) = -f(x) = -2/3$. For $k > 1$, we have

$$\begin{aligned} 0 &= f^{-1}f(x^k) = f^{-1}(x^{k-2})f(x^2) + f^{-1}(x^{k-1})f(x) + f^{-1}(x^k) \\ &= (2/3)^2 f^{-1}(x^{k-2}) + (2/3) f^{-1}(x^{k-1}) + f^{-1}(x^k). \end{aligned}$$

Hence $f^{-1}(x^k) = -(2/3)^2 f^{-1}(x^{k-2}) - (2/3) f^{-1}(x^{k-1})$.

Using this iteration formula we get,

$$|f^{-1}(x^k)| = \begin{cases} 0 & \text{if } k \equiv 2 \pmod{3}, \\ (2/3)^k & \text{if } k \not\equiv 2 \pmod{3}. \end{cases}$$

By Lemma 2, $f^{-1}(s) = 0$ for all $s \neq x^k$ for some k .

Now $\|f^{-1}\| < \sum_{k=0}^{\infty} (2/3)^k < \infty$; hence $f^{-1} \in A$, i.e. $0 \notin \text{Sp } f$. Nevertheless, $\sum_{s \neq e} |f(s)| = 10/9 > f(e)$.

In closing we mention that all the results above hold (with the same proof) if F was replaced by the free semigroup on n generators for an arbitrary n .

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