MULTIPLIERS AND DUALITY IN A^* -ALGEBRAS

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ABSTRACT. Let A be an A^* -algebra which is a dense $*$ -ideal of a B^* -algebra. Let $M_r(A)$ be the algebra of all bounded linear right multipliers on A . We obtain several characterizations of duality for A in terms of the weak operator topology on $M_r(A)$ and the embedding of $M_r(A)$ into the conjugate space of a Banach space.

1. Introduction. Let A be a (complex) Banach algebra. L. Máté has shown in [4] that if A has a weak right identity then $M_r(A)$ can be embedded anti-isomorphically into the second conjugate space A^{**} of A , when A^{**} is considered as a Banach algebra with Arens product. In [5] he has studied multipliers on Banach algebras which satisfy properties other than having a weak right identity. We state these properties in §3 and call them (P1) and (P2). Since every semisimple Banach algebra (and in particular every A^* -algebra) has property (P1), it is really property (P2) that plays an important role in this paper.

Let A be a Banach algebra. Máté [5] has constructed a Banach space Y' whose conjugate space Y'^* is a Banach algebra under a suitable product. If A has properties (P1) and (P2) then $M_r(A)$ can be embedded anti-isomorphically into Y'^* [5, Theorem 4, p. 232]. In particular, if A is a dual A^* -algebra which is a dense two-sided ideal of its completion, then A has property (P2) and $M_r(A)$ is anti-isomorphic to Y'^* . The converse also holds; namely, if A is an A^* -algebra such that A is a dense $*$ -ideal of a B^* -algebra, A has property (P2) and $M_r(A)$ is anti-isomorphic to Y'^* , then A is dual.

Let A be an A^* -algebra which is a dense $*$ -ideal of a B^* -algebra. Let τ_r be the weak operator topology on $M_r(A)$. Our main result states that A is dual if and only if A has property (P2) and $M_r(A)$ is τ_r -complete.

2. Notation and terminology. All algebras and vector spaces are over the complex field.

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Let A be a Banach algebra. A mapping T from A into itself is called a right multiplier if $T(xy) = xT(y)$, for all $x, y \in A$. Let $M_r(A)$ denote the set of all bounded linear right multipliers on A . $M_r(A)$ is a Banach algebra under the usual operations for operators and the operator bound norm.

For any set S in A , let $l(S)$ (resp. $r(S)$) denote the left (resp. right) annihilator of S in A , and let $\text{cl}(S)$ denote the closure of S in A . If, for every closed left ideal J and every closed right ideal R of A , we have $l(r(J)) = J$ and $r(l(R)) = R$, then A is called a dual algebra.

Let A be an A^* -algebra. The auxiliary norm on A will be denoted by $|\cdot|$. If A is dual then $|\cdot|$ is unique and the completion $\mathfrak{U}$ of A in this norm is a dual B^* -algebra [9, Lemma 5.5, p. 54]. For further details on A^* -algebras see [7].

If X is a Banach space then X^* and X^{**} will denote the conjugate and second conjugate spaces of X . For $x \in X$, $f \in X^*$ we shall also denote the value $f(x)$ by (x, f) . $\mathfrak{S}(X)$ will denote the closed unit ball of X . The canonical map of X into X^{**} will be denoted by π .

Let X be a Banach space and let B be an algebra of bounded linear operators on X into X . The weak operator topology on B is the topology on B generated by the seminorms $T \rightarrow |(T(x), f)|$, $x \in X$, $f \in X^*$. Under this topology B is a locally convex topological vector space in which multiplication is separately continuous. For any Banach algebra A , we shall denote the weak operator topology on $M_r(A)$ by τ_r .

3. Arens product and modified Arens product. Let A be a Banach algebra. Arens [1] has defined two products on A^{**} which make A^{**} into a Banach algebra. For completeness we sketch one of the Arens products that we shall need. This is done in stages as follows: Let $x, y \in A$, $f \in A^*$ and $F, G \in A^{**}$. Define $f * x \in A^*$ by $(f * x)y = f(xy)$. Define $F * f \in A^*$ by $(F * f)x = F(f * x)$. Define $F * G \in A^{**}$ by $(F * G)f = F(G * f)$. A^{**} is a Banach algebra under the product $F * G$ such that the canonical map of A into A^{**} is an isomorphism of A into $(A^{**}, *)$. (See also [3].)

Let Y be the linear hull of the set $\{f * x : f \in A^* \text{ and } x \in A\}$ and let $Y^\perp = \{F \in A^{**} : F(f * x) = 0 \text{ for all } f \in A^* \text{ and } x \in A\}$. Let Y' be the linear hull of the set

$$\left\{ b = \sum_{k=1}^{\infty} f_k * x_k : \sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty, f_k \in A^* \text{ and } x_k \in A \right\}.$$

By [5, Theorem 1, p. 229], Y' is a Banach space under the norm

$$\|b\|' = \inf \left\{ \sum_{k=1}^{\infty} \|f_k\| \|x_k\| : b = \sum_{k=1}^{\infty} f_k * x_k \right\}$$

and $\|b\| \leq \|b\|'$, for all $b \in Y'$. It is easy to see that if $F \in A^{**}$ then the restriction $F|Y'$ of F to Y' belongs to Y'^* and $\|F|Y'\|' \leq \|F\|$. Máté [5] has introduced a product in Y'^* (called the modified Arens product) which makes Y'^* into a Banach algebra. This product is defined as follows: Let $x, y \in A$, $h \in Y'$ and $F, G \in Y'^*$. Define $h * x \in Y'$ by $(h * x)y = h(xy)$ and define $F * h \in Y'$ by $(F * h)x = F(h * x)$. Define $F * G \in Y'^*$ by $(F * G)h = F(G * h)$. Then Y'^* is a Banach algebra under the product $F * G$.

For each $x \in A$, let $F_x \in Y'^*$ be given by $F_x(h) = h(x)$, for all $h \in Y'$. Then the mapping $x \rightarrow F_x$ is an algebraic isomorphism of A into $(Y'^*, *)$ [5, Proposition 2, p. 231]. Let $\pi'(A)$ denote the image of A in Y'^* by the mapping $x \rightarrow F_x$.

Máté [5] has considered Banach algebras A which have the following properties:

$$(P1) \quad \pi(A) \cap Y^\perp = (0).$$

(P2) $f_k \in A^*$, $x_k \in A$, $\sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty$ and $\sum_{k=1}^{\infty} f_k * x_k = 0$ implies that $\sum_{k=1}^{\infty} f_k(x_k) = 0$.

Lemma 3.1. *Let A be a dual A^* -algebra which is a two-sided ideal of its completion $\mathfrak{A}$. Then A has the properties (P1) and (P2).*

Proof. That A has property (P1) follows from the fact that A is semi-simple. (See the proof of [5, Lemma 1, p. 229].) Now suppose that there exist $f_k \in A^*$, $x_k \in A$,

$$\sum_{k=1}^{\infty} \|f_k\| \|x_k\| < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} f_k * x_k = 0.$$

Let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in A . Then

$$\sum_{k=1}^{\infty} f_k(x_k e_\alpha) = \sum_{k=1}^{\infty} (f_k * x_k) e_\alpha = 0,$$

for all α . Since every $a \in A$ can be expressed in the form $a = \sum_\alpha a e_\alpha$ [6, Theorem 16, p. 30], it is easy to see that $\sum_{k=1}^{\infty} f_k(x_k) = 0$. Hence A has property (P2).

4. Multipliers on dual A^* -algebras. We shall use the notation of the previous sections. Unless otherwise mentioned, A will be a dual A^* -algebra

which is a two-sided ideal of its completion $\mathfrak{U}$. Let I be the element of Y'^* given by

$$I(h) = \sum_{k=1}^{\infty} f_k(x_k),$$

where $h = \sum_{k=1}^{\infty} f_k * x_k \in Y'$. By Lemma 3.1, I is uniquely defined. We have $F * I = F$ for all $F \in Y'^*$ [5, p. 232] and clearly $\|I'\|' = 1$. By [5, Theorem 2, p. 231], every $T \in M_r(A)$ has a unique extension T' to all of Y'^* with $\|T'\|' \leq \|T\|$. Let $F^T = T'(I)$, for all $T \in M_r(A)$. Since A has properties (P1) and (P2), it follows that the mapping $T \rightarrow F^T$ is an anti-isomorphism of $M_r(A)$ into Y'^* such that $F_{T(x)} = T'(F_x) = F_x * F^T$, for $x \in A$ and $T \in M_r(A)$. (See the proof of [5, Theorem 3, p. 231].)

Lemma 4.1. $\pi'(A)$ is $\sigma(Y'^*, Y')$ -dense in Y'^* .

Proof. Since Y is dense in Y' , Lemma 3.1 implies that $\{h \in Y' : F_x(h) = 0, \text{ for all } x \in A\} = (0)$. Hence, by the bipolar theorem [8, 1.5, p. 126], $\pi'(A)$ is $\sigma(Y'^*, Y')$ -dense in Y'^* .

Lemma 4.2. For every $T \in M_r(A)$, we have

$$(T(a), f) = (f * a, F^T) \quad (a \in A, f \in A^*).$$

Proof. Let T^* denote the conjugate operator of T . Then $T^*(f * a) = T^*(f) * a$ [5, p. 231]. Hence

$$T'(I)(f * a) = I(T^*(f * a)) = I(T^*(f) * a) = (a, T^*(f)) = (T(a), f).$$

This completes the proof.

Now consider A as a left Banach A -module over itself. Then A^{**} may also be considered as a left Banach A -module with $aF = \pi(a) * F$, for all $a \in A$ and $F \in A^{**}$. Let $\text{Hom}_A(A, A^{**})$ be the set of all bounded linear maps T from A into A^{**} such that $T(ab) = aT(b)$ ($a, b \in A$).

Lemma 4.3. For each $F \in A^{**}$, the linear map $T_F : A \rightarrow A^{**}$ given by $(f, T_F(a)) = (f * a, F)$ belongs to $\text{Hom}_A(A, A^{**})$; moreover $T_F(a) = \pi(a) * F$, for all $a \in A$.

Proof. Since $\pi(A)$ is a right ideal of $(A^{**}, *)$ [11, Theorem 5.2, p. 830], $\pi(a) * F \in A^{**}$ for all $a \in A, F \in A^{**}$. But $F(f * a) = (F * f)a = (\pi(a) * F)f$. Hence $T_F(a) = \pi(a) * F$, for all $a \in A$; in particular $T_F(ab) = aT_F(b)$, for all $a, b \in A$. Clearly T_F is linear and continuous. Hence $T_F \in \text{Hom}_A(A, A^{**})$.

Corollary 4.4. *For each $F \in A^{**}$, there is a unique $T \in M_r(A)$ such that $\pi(T(a)) = T_F(a)$, for all $a \in A$.*

Proof. This follows from the fact that $M_r(A)$ and $M_r(\pi(A))$ are isometrically isomorphic and that $\pi(a) * F \in \pi(A)$ for all $a \in A$ and $F \in A^{**}$.

Lemma 4.5. *The mapping $T \rightarrow F^T$ is an isometric anti-isomorphism of $M_r(A)$ onto $(Y'^*, *)$.*

Proof. We have already observed that $T \rightarrow F^T$ is an anti-isomorphism of $M_r(A)$ into Y'^* . Now let $F \in Y'^*$ and let T_F be the bounded linear map on A into A^{**} given by $(f, T_F(a)) = (f * a, F)$, for all $a \in A, f \in A^*$. By Lemma 4.1, there exists a net $\{x_\alpha\}$ in A such that $F_{x_\alpha}(h) \rightarrow F(h)$, for all $h \in Y'$. But, by Corollary 4.4, for each α , there exists $T_\alpha \in M_r(A)$ such that $(f, T_\alpha(a)) = (f * a, F_{x_\alpha})$. Then $(f, T_\alpha(a)) \rightarrow (f, T_F(a))$, for all $a \in A, f \in A^*$, and consequently $T_F(ab) = aT_F(b)$, for all $a, b \in A$. Let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in A . We have $\sum_\alpha e_\alpha a = a$, for every $a \in A$. Hence $T_F(a) = T_F(\sum_\alpha e_\alpha a) = \sum_\alpha e_\alpha T_F(a)$, and since $e_\alpha T_F(a) \in \pi(A)$, for all e_α , it follows that $T_F(a) \in \pi(A)$, for all $a \in A$. Thus there exists a unique $T \in M_r(A)$ such that $\pi(T(a)) = T_F(a)$, for all $a \in A$. It is easy to see that $F = F^{T_F}$. Therefore $T \rightarrow F^T$ is onto Y'^* . To see that $F \rightarrow F^T$ is an isometry, we observe that $(T(a), f) = (f * a, F^T)$ gives $\|T\| \leq \|F^T\|'$. On the other hand, $\|F^T\|' = \|T'(I)\|' \leq \|T'\|' \|I\|' = \|T\|' \leq \|T\|$. (See the proof of [5, Theorem 2, p. 231].) Hence $\|F^T\|' = \|T\|$, for all $T \in M_r(A)$. This completes the proof.

As an application of Lemma 4.5, we have the following:

Theorem 4.6. *Let A be a minimal norm ideal of completely continuous operators on Hilbert space H . Then $M_r(A)$ is isometrically anti-isomorphic to $L(H)$, the algebra of continuous linear operators on H .*

Proof. Clearly A is a dense ideal of $LC(H)$, the algebra of compact linear operators on H , and from the proof of [9, Lemma 9.1, p. 64] it follows that A is a dual A^* -algebra. Let α be the given norm on A , $\mathfrak{K}$ the ideal of operators on H with finite dimensional range, $t(\cdot)$ and $\tau(\cdot)$ the trace and the trace norm on $\mathfrak{K}$. For $T \in L(H)$, let

$$\|T\|_\alpha = \sup \left\{ \frac{|t(XT)|}{\alpha(X)} : 0 \neq X \in \mathfrak{K} \right\}$$

and let $A_\alpha = \{T \in L(H) : \|T\|_\alpha < \infty\}$. A_α is a norm ideal, $\mathfrak{K} \subset A_\alpha$, and the map $\phi : f \rightarrow T_f$ is an isometric isomorphism of A^* onto A_α , where $f(T) = t(TT_f)$

for all $T \in A$, $f \in A^*$. (See [10, pp. 74–76].) We have $\phi(f * T) = T_f T$, $f \in A^*$, $T \in A$. Hence ϕ maps isomorphically Y into $\tau c(H)$, the algebra of trace-class operators on H , and, since every $S \in \mathfrak{K}$ is of the form $S = SE$ with E an idempotent in $\mathfrak{K}$, we have $\phi(Y) \supset \mathfrak{K}$. Since the norms α , $\|\cdot\|_\alpha$ are uniform crossnorms, $\|\cdot\|'$ is a uniform crossnorm on $\mathfrak{K}$ and therefore $\|S\|' \leq \tau(S)$, $S \in \mathfrak{K}$. But, for X, Z in $\mathfrak{K}$, we have $\tau(XZ) = t(W^*XZ) \leq \alpha(W^*X)\|Z\|_\alpha \leq \alpha(X)\|Z\|_\alpha$ [10, Theorem, p. 4], which gives $\tau(S) \leq \|S\|'$, $S \in \mathfrak{K}$. Thus $\|S\|' = \tau(S)$, $S \in \mathfrak{K}$. Therefore, since $\mathfrak{K}$ is dense in $\tau c(H)$ and Y is dense in Y' , it follows that Y' is isometrically isomorphic to $\tau c(H)$. An application of [10, Theorem 2, p. 47] and Lemma 4.5 completes the proof.

We are now ready to prove our main result.

Theorem 4.7. *Let A be an A^* -algebra which is a dense $*$ -ideal of a B^* -algebra $\mathfrak{U}$. Then the following statements are equivalent:*

- (i) A is dual.
- (ii) $M_r(A)$ is τ_r -complete and A has property (P2).

Proof. (i) $\Rightarrow$ (ii). Suppose A is dual. Then, by Lemma 3.1, A has property (P2) and, by Lemma 4.5, the mapping $T \rightarrow F^T$ is an isometric anti-isomorphism of $M_r(A)$ onto Y'^* . Since $\mathfrak{S}(Y'^*)$ is w^* -compact, it easily follows that $\mathfrak{S}(M_r(A))$ is τ_r -compact and hence that $M_r(A)$ is τ_r -complete.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Since A has property (P2), the mapping $T \rightarrow F^T$ is an isometric anti-isomorphism of $M_r(A)$ into Y'^* . Since A also has property (P1), Lemma 4.1 implies that $\pi'(A)$ is w^* -dense in Y'^* . For each $x \in A$, let T_x be the right multiplication operator on A given by x . The mapping $T \rightarrow F^T$ takes T_x into F_x . Thus the image of $M_r(A)$ by the map $T \rightarrow F^T$ is w^* -dense in Y'^* . Since $M_r(A)$ is τ_r -complete, it follows that the map $T \rightarrow F^T$ is onto Y'^* . In particular, $\mathfrak{S}(M_r(A))$ is τ_r -compact. Now let S be a bounded set in A and let T_s be the right multiplication operator on A given by s , $s \in S$. Let $x \in A$. Since $\mathfrak{S}(M_r(A))$ is τ_r -compact, the weak closure of the set $\{T_s(x) : s \in S\}$ in A is weakly compact. But $\{T_s(x) : s \in S\} = \{xs : s \in S\}$. Hence the left multiplication operator on A given by x is weakly completely continuous (w.c.c.), for every $x \in A$. Thus A is a left w.c.c. A^* -algebra. Since the involution in A is continuous, it is also right w.c.c. and consequently a w.c.c. A^* -algebra. Hence $\mathfrak{U}$ is w.c.c. [6, Lemma 9, p. 29] and therefore dual [6, Theorem 6, p. 21]. Let $\{e_\alpha\}$ be a maximal orthogonal family of selfadjoint minimal idempotents in A (and in $\mathfrak{U}$). Let T_α be the right multiplication operator on A given by e_α . For distinct $e_{\alpha_1}, \dots, e_{\alpha_n}$ form the sum $T_{\alpha_1} + \dots + T_{\alpha_n}$. These sums form a bounded set in $M_r(A)$ since

$$\begin{aligned} \|(T_{\alpha_1} + \cdots + T_{\alpha_n})a\| &= \|ae_{\alpha_1} + \cdots + ae_{\alpha_n}\| \\ &\leq k\|a\|\|e_{\alpha_1} + \cdots + e_{\alpha_n}\| = k\|a\|, \end{aligned}$$

for all $a \in A$. (Since A is a dense two-sided ideal of $\mathfrak{U}$, by [6, Lemma 4, p. 18] there exists a constant $k > 0$ such that $\|xy\| \leq k\|x\|\|y\|$, for all $x \in A$ and $y \in \mathfrak{U}$.) Since $\mathfrak{S}(M_r(A))$ is τ_r -compact, the set $\{(T_{\alpha_1} + \cdots + T_{\alpha_n})a\}$ has a weak limit point, say $a' \in A$. Clearly $ae_{\alpha} = a'e_{\alpha}$ for all α , so that $a = a'$. This shows that the (directed) set $\{ae_{\alpha_1} + \cdots + ae_{\alpha_n}\}$ converges weakly to a , for every $a \in A$. Similarly we can show that if the e_{α} are confined to any subfamily then $\{ae_{\alpha_1} + \cdots + ae_{\alpha_n}\}$ also converges weakly to an element of A . Therefore by Orlicz' Theorem [2, p. 240], $\sum_{\alpha} ae_{\alpha}$ is summable to a in the norm $\|\cdot\|$ of A . Thus $a \in \text{cl}(aA)$, for every $a \in A$, and consequently, by [6, Theorem 16, p. 30], A is dual. This completes the proof.

Corollary 4.8. *Let A be an A^* -algebra which is a dense $*$ -ideal of a B^* -algebra. Then A is dual if and only if the closed unit ball of $M_r(A)$ is τ_r -compact.*

Proof. This follows easily from the proof above.

Corollary 4.9. *Let A be an A^* -algebra which is a dense $*$ -ideal of a B^* -algebra. Then A is dual if and only if A has property (P2) and the mapping $T \rightarrow F^T$ takes $M_r(A)$ onto Y'^* .*

Proof. If A has property (P2), the proof of Lemma 4.5 shows that the mapping $T \rightarrow F^T$ is an isometry. Hence if $T \rightarrow F^T$ is onto then $M_r(A)$ is τ_r -complete.

Corollary 4.10. *Let A be an A^* -algebra which is a dense $*$ -ideal of a B^* -algebra. If A is reflexive, then A is dual.*

Proof. If A is reflexive then $\mathfrak{S}(A)$ is weakly compact and hence $\mathfrak{S}(M_r(A))$ is τ_r -compact.

Remark. Theorem 4.6 was originally stated for the algebra $\tau\mathcal{C}(H)$. I am grateful to the referee for pointing out that it is valid for all minimal norm ideals. The proof of Theorem 4.6 given here is a modification of the original proof.

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