MAXIMAL SINGULAR INTEGRAL TRANSFORMS ON LOCAL FIELDS

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ABSTRACT. We show that, on local fields, "nice" singular integral transforms preserve H^p -spaces for $0 < p < \infty$ where H^p is the space of all distributions whose maximal functions are in L^p . A version of the F. and M. Riesz theorem is also obtained.

1. By a local field we mean a locally compact, nondiscrete, totally disconnected (complete) field. Such a field is a p -adic field, a finite algebraic extension of a p -adic field, or a field of formal Laurent series over a finite field. The Calderón-Zygmund type of singular integral transforms on local fields were treated in [4] and [8]. The theory of H^p -spaces of conjugate systems in the sense of Stein-Weiss was studied in [2] and [1]. A more complete list of references of various results on harmonic analysis on local fields can be found in [1]. We shall assume some familiarities of notation and preliminaries in [1].

Let K be a fixed local field with the ring of integers $\mathcal{O}$. $\mathcal{O}/\mathcal{P} \cong GF(q)$ where $\mathcal{P}$ is the (unique) maximal ideal in $\mathcal{O}$ and q is a prime power. Let $\mathfrak{p}$ be a generator of $\mathcal{P}$, $|\mathfrak{p}| = q^{-1}$. For $k \in \mathbb{Z}$, let $\mathcal{P}^{-k} = \{x \in K: |x| \leq q^k\}$, ($\mathcal{O} = \mathcal{P}^0$). $\mathcal{P}_y^{-k} = y + \mathcal{P}^{-k}$ are spheres. The Haar measure on K has been normalized so that $|\mathcal{O}| = \int_{\mathcal{O}} dx = 1$ and $|\mathcal{P}_y^{-k}| = q^k$ for all k .

Let π be a (multiplicative) unitary character on K^* such that it is homogeneous of degree 0 and is ramified of degree $b \geq 1$ (i.e., $\pi(\mathfrak{p}^k x) = \pi(x)$ for all $k \in \mathbb{Z}$ and π is constant on $\mathcal{P}_1^b$ but not on $\mathcal{P}_1^{b-1}$). π is, thus, constant on cosets of $\mathcal{P}^{-m+b}$ in $\mathcal{P}^{-m} \setminus \mathcal{P}^{-m+1} = \{x \in K: |x| = q^m\}$ and $\int_{\{|x|=q^m\}} \pi(x) dx = 0$ for all $m \in \mathbb{Z}$. Denote $Q(x) = c_{\pi}(\pi(x)/|x|)$ where $c_{\pi} = 1/\Gamma(\pi)$. (See [5] for details.) Distributions on K have been identified with regular functions on $K \times \mathbb{Z}$. The regularization kernel $R_k(x) = q^{-k} \Phi_{-k}(x)$, where Φ_{-k} is the

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characteristic function of $\mathcal{P}^{-k}$, serves as the Poisson kernel. (See [7] and [1] for details.) It is easy to see that $R_n * Q = Q[1 - \Phi_{-(n+b-1)}]$. Let $Q_n = R_n * Q$ and $Q_n^N = Q_n \Phi_{-N}$ for $N \geq n + b$. We note that $Q_n^N * f(x, k) = Q_k^N * f(x, k) = Q^N * f(x, k)$ for $n \leq k \leq N - b$ if $f(x, k)$ is regular on $K \times \mathbb{Z}$.

For a distribution f on K (i.e., $f(x, k) = (R_k * f)(x)$, a regular function on $K \times \mathbb{Z}$), define

$$(T_\pi f)(x, k) = \lim_{N \rightarrow \infty} Q^N * f(x, k) \quad \text{for } (x, k) \in K \times \mathbb{Z}.$$

For "nice" functions f on K , this is just a sort of singular integral transform as was studied in [4], [8] and [1]. In particular, if $f \in L^p(K)$ with $1 \leq p < \infty$, then $(T_\pi f)(x, k) = T_\pi(f(x, k))$ is a well-defined regular function on $K \times \mathbb{Z}$ and $T_\pi f(x) = \lim_{k \rightarrow -\infty} (T_\pi f)(x, k)$ exists almost everywhere. Moreover T_π is of weak-type (1,1) and of type (p, p) for $1 < p < \infty$. It is obvious that the maximal operator

$$f \rightarrow f^*(x) = \sup_{k \in \mathbb{Z}} |f(x, k)| = \sup_{k \in \mathbb{Z}} |R_k * f(x)|$$

as well as the Hardy-Littlewood maximal operator $f \rightarrow Mf(x) = \sup_{k \in \mathbb{Z}} R_k * |f|(x)$ is also of weak-type (1,1) and of type (p, p) for $1 < p < \infty$.

For $0 < p < \infty$, let $H^p(K)$ be the space of all distributions f on K whose maximal function $f^* \in L^p(K)$ with the "norm" $\|f\|_{H^p} = \|f^*\|_p$. This way of defining H^p is suggested by the results of Burkholder-Gundy-Silverstein and Fefferman-Stein (see e.g. [3]). Hence, $H^p = L^p$ if $1 < p < \infty$ and $H^1 \subset L^1$. Indeed, if $1 < p < \infty$, since $f = T_{\pi^{-1}}(T_\pi f)$, we have

$$A_p \|f\|_p \leq \|T_\pi f\|_p \leq B_p \|f\|_p \quad \text{with constants } A_p, B_p > 0$$

and

$$\|f\|_p \leq \|f^*\|_p \leq p' \|f\|_p \quad \text{where } 1/p + 1/p' = 1.$$

In other words, $\|f^*\|_p \approx \|f\|_p \approx \|T_\pi f\|_p \approx \|(T_\pi f)^*\|_p$ if $1 < p < \infty$. Our main result is to show that T_π preserves H^p for $0 < p < \infty$. That is, for $0 < p < \infty$,

$$\|(T_\pi f)^*\|_p \leq C_p \|f^*\|_p \quad \text{for some } C_p > 0$$

and hence

$$\|f^*\|_p \approx \|(T_\pi f)^*\|_p \quad \text{for } 0 < p < \infty.$$

A sharper version of the F. and M. Riesz theorem on local fields (than that of [1]) is obtained as a consequence.

2. We have only to consider the case $0 < p \leq 1$. Let $f \in H^p$, i.e., $f(x, k)$ is a regular function on $K \times \mathbf{Z}$ such that $f^*(x) = \sup_k |f(x, k)| \in L^p$. We first show that $(T_\pi f)(x, k)$ exists for all x and all k .

For $\lambda > 0$, the set $\{x \in K : f^*(x) > \lambda\}$ can be written as the disjoint union of spheres $\{\mathcal{P}_{x_j}^{-(l_j-1)}\}_{j=1}^\infty$ such that $|f(x_j, l_j - 1)| > \lambda \geq |f(x_j, l_j)|$. For a sphere $S = \mathcal{P}_x^{-l}$, let $S^h = \mathcal{P}_x^{-(l+h-1)}$. Let $D_\lambda = \bigcup_{j=1}^\infty \mathcal{P}_{x_j}^{-l_j} = \bigcup_{t=1}^\infty S_t$ where the spheres $S_t = \mathcal{P}_{x_j}^{-l_j}$, $t = 1, 2, \dots$, are mutually disjoint, and $D_\lambda^h = \bigcup_{t=1}^\infty S_t^h$. Hence,

$$(1) \quad |D_\lambda| \leq \sum_{t=1}^\infty |S_t| = q|\{f^* > \lambda\}| \quad \text{and} \quad |D_\lambda^h| \leq q^{h-1}|D_\lambda|.$$

For a fixed $k \in \mathbf{Z}$, let $D_{\lambda k} = \bigcup_t S_{tk}$ where $\{S_{tk}\}$ are those spheres $S_t = \mathcal{P}_{x_j}^{-l_j}$ such that $l_j > k$, and $D_{\lambda k}^h = \bigcup_t S_{tk}^h$.

Let $g(x, k) = f(x_j, l_j)$ if $\mathcal{P}_x^{-k} \subset S_t = \mathcal{P}_{x_j}^{-l_j}$ and $g(x, k) = f(x, k)$ otherwise. Then $g(x, k)$ is regular on $K \times \mathbf{Z}$, $|g(x, k)| \leq \lambda$ and $|g(x, k)| \leq f^*(x) \in L^p$. Thus $g(x, k) \in L^2$ for all k . Moreover, $g(x) = \lim_{k \rightarrow -\infty} g(x, k)$ exists a.e. and is in L^2 . Write $b(x, k) = f(x, k) - g(x, k)$ for all x and all k . We have that $\int_{S_{tk}} b(x, k) dx = 0$ for all S_{tk} in $D_{\lambda k}$, also that $b(x, k) = 0$ if $\mathcal{P}_x^{-k} \not\subset D_{\lambda k}$, i.e., $x \notin D_{\lambda k}$.

Now, for fixed $k \in \mathbf{Z}$ and for large N such that $N \geq k + h$ and $|D_\lambda| < q^N$,

$$\begin{aligned} Q^N * b(x, k) &= \int_{\mathcal{P}^{-N}} Q_k(y) b(x - y, k) dy \\ &= \int_{\mathcal{P}_x^{-N} \cap D_{\lambda k}^h} Q_k(x - y) b(y, k) dy = \sum \int_{S_{tk}} Q_k(x - y) b(y, k) dy \end{aligned}$$

where the summation is taken over all spheres S_{tk} in $D_{\lambda k}$ such that $S_{tk} \subset \mathcal{P}_x^{-N}$. (Note that the sphere $\mathcal{P}_x^{-N}$ is larger than any sphere S_{tk} in $D_{\lambda k}$, so if $S_{tk} \cap \mathcal{P}_x^{-N} \neq \emptyset$, then $S_{tk} \subset \mathcal{P}_x^{-N}$.) If $x \notin D_{\lambda k}^h$ and $y \in S_{tk} = \mathcal{P}_{x_t}^{-l_t}$, then $|x - x_t| = q^m \geq q^{l_t+h}$ and $|(x - y) - (x - x_t)| = |y - x_t| \leq q^{l_t} \leq q^{m-h}$. Since Q_k is constant on cosets of $\mathcal{P}^{-m+h}$ in $\{x : |x| = q^m\}$, we see that $Q_k(x - y) = Q_k(x - x_t)$. From the fact that $\int_{S_{tk}} b(y, k) dy = 0$, it follows that $Q^N * b(x, k) = 0$ for large N and if $x \notin D_{\lambda k}^h \subset D_\lambda^h$.

Since $g(x, k) \in L^2$ for all k , $Q^N * g(x, k)$ is well defined for all x so is $(T_\pi g)(x, k)$. Moreover, $(T_\pi g)(x, k) \in L^2$ for all k . Hence, for any fixed k , $Q^N * f(x, k) = Q^N * g(x, k) + Q^N * b(x, k)$ is defined for all large N and for all x , except possibly for $x \in D_{\lambda k}^h$. Since $f^* \in L^p$, we have $|D_\lambda| \rightarrow 0$ as $\lambda \rightarrow \infty$. Hence $|D_{\lambda k}^h| = 0$ for sufficiently large λ . Therefore $(T_\pi f)(x, k) = \lim_{N \rightarrow \infty} Q^N * f(x, k)$ is defined for all x .

It is easy to see that $(T_\pi f)(x, k)$ is a regular function on $K \times \mathbb{Z}$. Call the corresponding distribution on K , $T_\pi f$. We note that, by comparing the Fourier transforms, the regularization of the distribution $T_\pi f$ is just $(T_\pi f)^*(x, k)$. It remains to show that $(T_\pi f)^* \in L^p$.

Lemma. For $\lambda > 0$, let $E_\lambda = \{x \in K : (T_\pi f)^*(x) > 2\lambda\}$.

$$(2) \quad |E_\lambda| \leq A\lambda^{-2} \int_0^\lambda |D_s| s \, ds \quad \text{for some constant } A > 0.$$

Proof. Write $f(x, k) = g(x, k) + b(x, k)$ as above. Since

$$\begin{aligned} (T_\pi f)^*(x) &= \sup_k |(T_\pi f)(x, k)| \\ &\leq \sup_k |(T_\pi g)(x, k)| + \sup_k |(T_\pi b)(x, k)| = (T_\pi g)^*(x) + (T_\pi b)^*(x), \end{aligned}$$

we have

$$(3) \quad |E_\lambda| = |\{(T_\pi f)^* > 2\lambda\}| \leq |\{(T_\pi g)^* > \lambda\}| + |\{(T_\pi b)^* > \lambda\}|.$$

It follows from the L^2 -theory, since $g(x) = \lim_{k \rightarrow \infty} g(x, k) \in L^2$, that

$$|\{(T_\pi g)^* > \lambda\}| \leq 2\lambda^{-2} \|T_\pi g\|_2^2 \leq 2B\lambda^{-2} \|g\|_2^2 \leq 2B\lambda^{-2} \int_{x \notin D_\lambda} (f^*(x))^2 \, dx + 2B|D_\lambda|.$$

Moreover,

$$\begin{aligned} \lambda^{-2} \int_{x \notin D_\lambda} (f^*(x))^2 \, dx &\leq \lambda^{-2} \int_{\{f^* \leq \lambda\}} (f^*(x))^2 \, dx \\ &= \lambda^{-2} \int_0^\lambda |\{f^* > s\}| 2s \, ds \leq 2\lambda^{-2} \int_0^\lambda |D_s| s \, ds, \end{aligned}$$

and

$$|D_\lambda| = |D_\lambda| 2\lambda^{-2} \int_0^\lambda s \, ds \leq 2\lambda^{-2} \int_0^\lambda |D_s| s \, ds.$$

Hence, we have

$$(4) \quad |\{(T_\pi g)^* > \lambda\}| \leq 4B\lambda^{-2} \int_0^\lambda |D_s| s \, ds.$$

We have shown that $(T_\pi b)(x, k) = 0$ if $x \notin D_{\lambda k}^b \subset D_\lambda^b$. Thus, $(T_\pi b)^*(x) = \sup_k |(T_\pi b)(x, k)| = 0$ if $x \notin D^b$. Therefore

$$(5) \quad |\{(T_\pi b)^* > \lambda\}| \leq |D_\lambda^b| \leq q^{b-1} |D_\lambda| \leq q^{b-1} 2\lambda^{-2} \int_0^\lambda |D_s| s \, ds.$$

(3), (4) and (5) give us the estimate (2). The Lemma is proved.

Now, from (2) and (1), we have $|E_\lambda| \leq Aq\lambda^{-2} \int_0^\lambda |\{f^* > s\}| s \, ds$. Thus, by a usual integration argument with $0 < p \leq 1$,

$$\begin{aligned}
\int_0^\infty \lambda^{p-1} |E_\lambda| d\lambda &\leq Aq \int_0^\infty \lambda^{p-3} \int_0^\lambda |\{f^* > s\}| s ds d\lambda \\
&= Aq \int_0^\infty \int_s^\infty \lambda^{p-3} d\lambda |\{f^* > s\}| s ds \\
&= \frac{Aq}{2-p} \int_0^\infty s^{p-1} |\{f^* > s\}| ds = \frac{Aq}{p(2-p)} \|f^*\|_p^p.
\end{aligned}$$

Hence, $\|(T_\pi f)^*\|_p \leq C_p \|f^*\|_p$ where C_p depends only on π, q and p .

Therefore we have shown the following

Theorem. For $0 < p < \infty$, $f \in H^p$ implies $T_\pi f \in H^p$ and, moreover, $\|f\|_{H^p} \approx \|T_\pi f\|_{H^p}$.

Remark. With some slight modifications the above argument also applies to "nice" singular integral transforms on K^n , the n -dimensional vector space over K , and to multiplier transforms T on K^n such that $(Tf)^\wedge = m\hat{f}$ with m satisfying some smooth conditions, in particular, those cases studied in [8, §4].

3. Let us consider the most interesting case, $p = 1$. The following result is a (sharper) version of the F. and M. Riesz theorem:

Theorem. Suppose $\mu_0, \mu_1, \dots, \mu_m$ are finite Borel measures on K . If $F(x, k) = (\mu_0(x, k), \mu_1(x, k), \dots, \mu_m(x, k))$ forms a "conjugate system", then each μ_j is absolutely continuous and $\mu_j^* \in L^1$, $j = 0, 1, \dots, m$. Moreover, $T_\pi \mu_j$, $j = 0, 1, \dots, m$, is also absolutely continuous for any "nice" singular integral transform T_π (as described earlier).

Proof. The first part of the theorem was proved in [1]. From our main result, since $\mu_j^* \in L^1$, we have $(T_\pi \mu_j)^* \in L^1$. Hence, $(T_\pi \mu_j)(x, k)$ is the regularization of an L^1 -function. Therefore, $T_\pi \mu_j$ is absolutely continuous.

The set $\bigcup_{k=-\infty}^\infty p^k(\epsilon + \mathcal{P})$, where $\epsilon + \mathcal{P}$ is a coset of $\mathcal{P}$ in $\{|x| = 1\}$, is said to be a "cone". As a corollary, we have another version of the Riesz brothers' theorem:

Theorem. Suppose μ is a finite Borel measure on K such that $\hat{\mu}$, the Fourier transform of μ , is supported on a "cone". Then μ is absolutely continuous, so is $T_\pi \mu$.

As a final remark we consider the case $p = 1$ for the Hardy-Littlewood maximal operator $f \rightarrow Mf = \sup_k R_k * |f|$. We show the following local field variant of a result of Stein [6]:

Theorem. Suppose f is an L^1 -function which is supported on some sphere S . Then $\int_S Mf(x) dx < \infty$ if and only if $\int_S |f(x)| \log^+ |f(x)| dx < \infty$.

Proof. The fact that $\int_S |f| \log^+ |f| dx < \infty$ implies $\int_S Mf dx < \infty$ is known (e.g., [4]) just as its well-known analogue in the $\mathbb{R}^n$ setting. With some modifications, the converse can be similarly shown.

For $\lambda > 0$, let $F_\lambda = \{x : Mf(x) > \lambda\}$. Write $F_\lambda = \bigcup_{t=1}^\infty S_t$ where $\{S_t\}_{t=1}^\infty$ are mutually disjoint spheres such that

$$\frac{1}{|S_t|} \int_{S_t} |f(x)| dx > \lambda \geq \frac{1}{|S_t^2|} \int_{S_t^2} |f(x)| dx.$$

Let $A = \int_S |f| dx$. For $x \notin S$, since f is supported on S , we have $Mf(x) < A$. Thus, if $\int_S Mf dx < \infty$, then $\int_B Mf dx < \infty$ for any finite sphere B . Hence, $\int_{F_\alpha} Mf dx < \infty$ for all $\alpha > 0$. Consequently, $\int_\alpha^\infty |F_\lambda| d\lambda < \infty$ for any $\alpha > 0$. Now,

$$|F_\lambda| = \sum_t |S_t| = q^{-1} \sum_t |S_t^2| \geq q^{-1} \lambda^{-1} \sum_t \int_{S_t^2} |f| dx \geq q^{-1} \lambda^{-1} \int_{F_\lambda} |f| dx.$$

Therefore, by a usual integration argument, it follows that $\int_S |f| \log^+ |f| dx < \infty$.

We note that $Mf \in L^1(K)$ implies $f \in L \log^+ L(K)$, while the converse implication fails.

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