REPLACING CONVEX SETS BY POLYTOPES

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ABSTRACT. A conjecture of A. J. Hoffman is settled by showing that if P is a d -polytope in E^d , if compact convex subsets $C_1, \dots, C_k$ are such that every t -dimensional affine flat that meets P also meets $\bigcup_{i=1}^k C_i$, then there exist polytopes $D_1, \dots, D_k$, with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every t -flat that meets P also meets $\bigcup_{i=1}^k D_i$, provided $k \leq d - t + 1$. Counterexamples are given for the cases where $d \geq 3$, $1 \leq t \leq d - 2$ and $k \geq d - t + 2$.

A. J. Hoffman [4] proved that if P is a d -polytope and if $C_1, \dots, C_k$ are compact convex subsets of P , the union of which covers P , then there exist polytopes $D_1, \dots, D_k$, with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that $P = \bigcup_{i=1}^k D_i$; he then stated the following

Conjecture (d, t, k). If P is a d -polytope in E^d , $C_1, \dots, C_k$ are compact convex subsets of P , $t \geq 0$ is an integer such that every affine t -flat that meets P also meets $\bigcup_{i=1}^k C_i$, then there exist polytopes $D_1, \dots, D_k$ with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every affine t -flat that meets P also meets $\bigcup_{i=1}^k D_i$.

Obviously, Hoffman's theorem settles Conjecture ($d, 0, k$) for all $d \geq 1$ and all $k \geq 1$. J. Zaks [6] gave counterexamples to Conjecture ($d, d-2, 4$), for all $d \geq 3$; this implies that Conjecture ($d, d-2, k$) is false for all $d \geq 3$ and $k \geq 4$. J. Zaks [7] proved Conjecture ($d, d-1, k$) for all $d \geq 2$ and $k \geq 0$; a simpler proof of this was given by H. G. Eggleston [1], where the dual form is treated. Conjecture ($d, t, 1$) is clearly true, while Conjecture ($d, t, 2$) was proved by W. R. Hare and C. R. Smith [3] for all t satisfying $0 \leq t \leq d - 1$.

The purpose of this paper is to prove Conjecture (d, t, k) for all $d \geq 1$, all t satisfying $1 \leq t < d - 1$ and all k satisfying $1 \leq k \leq d - t + 1$; in addition, counterexamples will be given here to Conjecture ($d, t, d - t + 2$), for all $d \geq 3$ and all t satisfying $1 \leq t \leq d - 2$; this implies that Conjecture

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(d, t, k) is false for all $d \geq 3$, all t satisfying $1 \leq t \leq d - 2$ and all k satisfying $k \geq d - t + 2$. Consequently, Conjecture (d, t, k) is settled for all d, t and k .

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Throughout the paper we shall follow the terminology of B. Grünbaum [2]; the j -skeleton of a polytope P , denoted by $\text{skel}_j P$, is the union of all the faces of P of dimension at most j .

We need the following few lemmas.

Lemma 1 (Compare with Lemma 1 of [3]). *If P is a d -polytope in E^d and C is a subset of P , such that every t -flat that meets P also meets C , then $\text{skel}_{d-t-1} P \subseteq C$.*

Proof. Let $x \in \text{skel}_{d-t-1} P$; there exists a hyperplane H in E^d , such that $H \cap P$ is a face of P of dimension at most $d - t - 1$ which contains x . The dimension of $\text{aff}(H \cap P)$ is at most $d - t - 1$, hence H contains a t -flat L , such that $L \cap \text{aff}(H \cap P) = \{x\}$; hence $L \cap P = \{x\}$, and it follows from the condition of the lemma that $x \in C$, as required.

Lemma 2. *If P is a d -polytope in E^d , and $C_1, \dots, C_k$ are compact convex subsets of P such that every t -flat that meets P also meets $\bigcup_{i=1}^k C_i$, and if $\bigcap_{i=1}^k C_i \neq \emptyset$, then there exist polytopes $D_1, \dots, D_k$ with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every t -flat that meets P also meets $\bigcup_{i=1}^k D_i$.*

Proof. Let $F_1, \dots, F_m$ be the $(d - t - 1)$ -faces of P . It follows from Lemma 1 that $\text{skel}_{d-t-1} P \subseteq \bigcup_{i=1}^k C_i$; hence F_j is covered by $\bigcup_{i=1}^k (C_i \cap F_j)$. By Hoffman's theorem, applied to each one of the $(d - t - 1)$ -polytopes $F_1, \dots, F_m$, it follows that there exist polytopes D_{ij} , for all i and j , $1 \leq i \leq k$ and $1 \leq j \leq m$, such that

$$(1) D_{ij} \subseteq C_i \cap F_j \text{ and}$$

$$(2) F_j = \bigcup_{i=1}^k D_{ij}.$$

Let $x \in \bigcap_{i=1}^k C_i$, and define D_i by $D_i = \text{conv}(\{x\} \cup \bigcup_{j=1}^m D_{ij})$, for all $1 \leq i \leq k$. Clearly, D_i is a polytope in C_i , for all $1 \leq i \leq k$;

Claim. Every t -flat that meets P also meets $\bigcup_{i=1}^k D_i$.

To show this, let L be a t -flat in E^d that meets P , and let M be a $(d - t)$ -flat in E^d , orthogonal to L . Define the set Q by $Q = \{\lambda x + (1 - \lambda)y \mid y \in \text{skel}_{d-t-1} P, 0 \leq \lambda \leq 1\}$; clearly $Q \subseteq \bigcup_{i=1}^k D_i$, hence it suffices to show that L meets Q .

Let $\text{proj}_M: E^d \rightarrow M$ be the orthogonal projection of E^d onto M ; as is well known [5], see also [2, p. 74, Theorem 8], $\text{proj}_M(P)$ is a polytope of dimension $d - t$ ($= \dim M$), hence

$$\text{rel bd proj}_M(P) \subseteq \text{proj}_M(\text{skel}_{d-t-1} P).$$

Since $Q \supseteq \text{skel}_{d-t-1} P$, it follows that $\text{proj}_M(P) = \text{proj}_M(Q)$; $\text{proj}_M(L)$ is a point, since $L \perp M$, and $\text{proj}_M(L) \in \text{proj}_M(P)$, because L meets P ; therefore $\text{proj}_M(L) \in \text{proj}_M(Q)$. It follows now, by taking the inverse of proj_M , that L meets Q .

This completes the proof of Lemma 2.

Lemma 3. *If $C_1, \dots, C_{d-t+1}$ are compact convex sets in E^d , such that $\bigcap_{i=1}^{d-t+1} C_i = \emptyset$, and if $0 < t \leq d - 1$ and every collection of $d - t$ of the sets $C_1, \dots, C_{d-t+1}$ has a nonempty intersection, then there exists a t -flat L in E^d such that*

- (1) $L \cap (\bigcup_{i=1}^{d-t+1} C_i) = \emptyset$ and
- (2) $L \cap \text{conv}(\bigcup_{i=1}^{d-t+1} C_i) \neq \emptyset$.

Proof. The proof is by induction on $k = d - t + 1$; in case $k = 2$, it follows that $t = d - 1$ and that the two compact convex sets C_1 and C_2 are disjoint; the required t -flat L is any of the hyperplanes that strictly separates C_1 from C_2 .

Assume inductively that the statement is true for $k - 1$ ($= d - t$), for $k > 2$, and let $C_1, \dots, C_{d-t+1}$ be compact convex sets in E^d , as described in the lemma.

C_1 and $\bigcap_{i=2}^{d-t+1} C_i$ are two nonempty disjoint compact convex sets, hence there exists a hyperplane H that strictly separates them, i.e. $C_1 \subset H^-$ and $\bigcap_{i=2}^{d-t+1} C_i \subset H^+$, where H^- and H^+ are the two open half-spaces determined by H .

Define C_i^* by $C_i^* = C_i \cap H$ for all $2 \leq i \leq d - t + 1$. Clearly, C_i^* is a compact convex set in the hyperplane H , and since $\bigcap_{i=2}^{d-t+1} C_i \subset H^+$, it follows that $\bigcap_{i=2}^{d-t+1} C_i^* = \emptyset$.

Claim. Every collection of $d - t - 1$ of the sets $C_2^*, \dots, C_{d-t+1}^*$ has a nonempty intersection.

Let T^* be such an intersection, and let T be the corresponding intersection of the $d - t - 1$ of the sets $C_2, \dots, C_{d-t+1}$; clearly, $T^* = T \cap H$. $T \cap C_1$ is an intersection of $d - t$ of the sets $C_1, \dots, C_{d-t+1}$, hence by the conditions of the lemma it follows that $T \cap C_1 \neq \emptyset$. Therefore $T \neq \emptyset$; since $\bigcap_{i=2}^{d-t+1} C_i \subset H^+$ and T is the intersection of $d - t - 1$ of the sets

$C_2, \dots, C_{d-t+1}$, it follows that $T \cap H^+ \neq \emptyset$. Both $C_1 \subset H^-$ and $T \cap C_1 \neq \emptyset$ imply that $T \cap H^- \neq \emptyset$. Since T is a convex set, it follows from both $T \cap H^+ \neq \emptyset$ and $T \cap H^- \neq \emptyset$ that $T \cap H \neq \emptyset$; therefore $T^* = T \cap H \neq \emptyset$, and the proof of the claim is completed.

As a result, the inductive assumption is applicable to the sets $C_2^*, \dots, C_{d-t+1}^*$ in the hyperplane H , implying the existence of a t -flat L in H , such that

$$L \cap \left(\bigcup_{i=2}^{d-t+1} C_i^* \right) = \emptyset \quad \text{and} \quad L \cap \text{conv} \left(\bigcup_{i=2}^{d-t+1} C_i^* \right) \neq \emptyset.$$

Since $C_1 \subset H^-$, $L \subset H$ and $C_i^* = C_i \cap H$, it follows that $L \cap (\bigcup_{i=1}^{d-t+1} C_i) = \emptyset$.

$$L \cap \text{conv} \left(\bigcup_{i=1}^{d-t+1} C_i \right) \supseteq L \cap \text{conv} \left(\bigcup_{i=2}^{d-t+1} C_i \right) \supseteq L \cap \text{conv} \left(\bigcup_{i=2}^{d-t+1} C_i^* \right) \neq \emptyset,$$

therefore $L \cap \text{conv}(\bigcup_{i=1}^{d-t+1} C_i) \neq \emptyset$, which completes the proof of Lemma 3.

Lemma 4. *If P is a polytope in E^d and if $C_1, \dots, C_{d-t+1}$ are compact convex subsets of P , such that*

$$(1) \quad P \neq \bigcup_{i=1}^{d-t+1} C_i \quad \text{and} \quad \bigcap_{i=1}^{d-t+1} C_i = \emptyset,$$

then there exists a t -flat that meets P and does not meet $\bigcup_{i=1}^{d-t+1} C_i$.

Proof. The proof is by induction on $k = d - t + 1$. In case $k = 2$, $t = d - 1$ and the two compact convex subsets C_1 and C_2 of P are disjoint; if both of C_1 and C_2 are nonempty, then they are strictly separated by a hyperplane H , where H meets P and does not meet $C_1 \cup C_2$, as required. If $C_1 = \emptyset$ and $C_2 \neq \emptyset$, then $P \neq C_2$ implies that at least one vertex v of P is not in C_2 ; any hyperplane that supports P at v has the property that it meets P and does not meet $C_1 \cup C_2$. The case where $C_1 = C_2 = \emptyset$ is trivial.

Assume inductively that the statement is true for $k - 1$, $k > 2$, and let P be a polytope in E^d , $C_1, \dots, C_{d-t+1}$ be compact convex subsets of P , where $d - t + 1 = k$, such that (1) is true.

If every collection of $d - t$ sets among the sets $C_1, \dots, C_{d-t+1}$ has a nonempty intersection, then the result follows by Lemma 3. Otherwise, some $d - t$ of the sets among $C_1, \dots, C_{d-t+1}$ have nonempty intersection, say $\bigcap_{i=1}^{d-t} C_i \neq \emptyset$.

Since $P \neq \bigcup_{i=1}^{d-t+1} C_i$, let $x \in P \setminus \bigcup_{i=1}^{d-t+1} C_i$; $x \notin C_{d-t+1}$, hence there exists a hyperplane H that contains x and does not meet C_{d-t+1} . Define $P^*, C_1^*, \dots, C_{d-t}^*$ by $P^* = P \cap H$ and $C_i^* = C_i \cap H$ for all $1 \leq i \leq d-t$. P^* is a polytope in the $(d-1)$ -flat H , $C_1^*, \dots, C_{d-t}^*$ are compact convex subsets of P^* , such that

$$P^* \neq \bigcup_{i=1}^{d-t} C_i^* \quad \text{and} \quad \bigcap_{i=1}^{d-t} C_i^* = \emptyset;$$

it follows by induction that H contains a t -flat L that meets P^* and does not meet $\bigcup_{i=1}^{d-t} C_i^*$. L therefore does not meet $\bigcup_{i=1}^{d-t} C_i$, and since $H \cap C_{d-t+1} = \emptyset$ and $L \subset H$, L does not meet C_{d-t+1} ; therefore $L \cap (\bigcup_{i=1}^{d-t+1} C_i) = \emptyset$. Clearly L meets P , since it meets P^* and $P^* \subset P$.

This completes the proof of Lemma 4.

We are ready for the proof of our main result.

Theorem 1. *If P is a polytope in E^d and $C_1, \dots, C_k$ are compact convex subsets of P , such that every t -flat that meets P also meets $\bigcup_{i=1}^k C_i$, and if $1 \leq t \leq d-2$ and $1 \leq k \leq d-t+1$, then there exist polytopes $D_1, \dots, D_k$, with $D_i \subseteq C_i$ for all $1 \leq i \leq k$, such that every t -flat that meets P also meets $\bigcup_{i=1}^k D_i$.*

Theorem 1 states, of course, that Conjecture (d, t, k) is true for all $d \geq 3$, $1 \leq t \leq d-2$ and $1 \leq k \leq d-t+1$.

Proof of Theorem 1. If $k < d-t+1$, then define $C_i = \emptyset$ for all $k < i \leq d-t+1$. If $\bigcap_{i=1}^{d-t+1} C_i \neq \emptyset$, then the proof follows from Lemma 2.

If $\bigcap_{i=1}^{d-t+1} C_i = \emptyset$, then since every t -flat that meets P also meets $\bigcup_{i=1}^{d-t+1} C_i$, it follows from Lemma 4 that $P = \bigcup_{i=1}^{d-t+1} C_i$. There exist, by Hoffman's theorem, polytopes $D_1, \dots, D_{d-t+1}$, with $D_i \subseteq C_i$ for all $1 \leq i \leq d-t+1$, such that $P = \bigcup_{i=1}^{d-t+1} D_i$; it follows that every t -flat that meets P meets $\bigcup_{i=1}^{d-t+1} D_i$, and the proof of Theorem 1 is complete.

As for the remaining cases of Conjecture (d, t, k) , we have

Theorem 2. *Conjecture (d, t, k) is false for all $d \geq 3$, $1 \leq t \leq d-2$ and $k \geq d-t+2$.*

Proof. It is certainly sufficient to prove that Conjecture (d, t, k) is false for all $d \geq 3$, $1 \leq t \leq d-2$ and $k = d-t+2$.

Case 1. $t = 1$. The proof is by induction on d , starting with the counter-example to Conjecture $(3, 1, 4)$, given in [6].

Suppose P is a polytope in E^d and $C_1, \dots, C_{d+1}$ are compact convex subsets of P , which provide a counterexample to Conjecture $(d, 1, d+1)$. Construct a counterexample to Conjecture $(d+1, 1, d+2)$ as follows: regard E^d as a subspace of E^{d+1} in the usual way, take $x \in E^{d+1} \setminus \text{aff}(P)$, and define

$$P^* = \text{conv}(\{x\} \cup P),$$

$$C_i^* = \text{conv}(\{x\} \cup C_i), \text{ for all } i, 1 \leq i \leq d+1, \text{ and}$$

$$C_{d+2}^* = P.$$

To show that $P^*, C_1^*, \dots, C_{d+2}^*$ provide a counterexample to Conjecture $(d+1, 1, d+2)$, let L be any line in E^{d+1} that meets P^* . If L meets x or P , then L certainly meets $\bigcup_{i=1}^{d+2} C_i^*$. Otherwise, L does not meet $\{x\} \cup P$; since L meets P^* , L meets some section of P^* (= an intersection of P^* with a hyperplane that strictly separates x and P). This section of P^* , together with the corresponding sections of $C_1^*, \dots, C_{d+1}^*$, is projectively equivalent to the original counterexample to Conjecture $(d, 1, d+1)$; therefore L meets the union of the sections of $C_1^*, \dots, C_{d+1}^*$, hence L meets $\bigcup_{i=1}^{d+2} C_i^*$.

Suppose $D_1^*, \dots, D_{d+2}^*$ are polytopes, with $D_i^* \subseteq C_i^*$ for all $1 \leq i \leq d+2$. Let H be a hyperplane, parallel to $\text{aff}(P)$, in E^{d+1} that meets P^* and misses $\{x\} \cup P$.

Considering the polytope $P^* \cap H$ with the collection of polytopal subsets $D_1^* \cap H, D_2^* \cap H, \dots, D_{d+1}^* \cap H$ of $P^* \cap H$, and using the obvious projective equivalence of them with the original counterexample to Conjecture $(d, 1, d+1)$, it follows that H contains a line L that meets P^* and does not meet $\bigcup_{i=1}^{d+1} (D_i^* \cap H)$. It follows that $L \cap P^* \neq \emptyset$ and $L \cap (\bigcup_{i=1}^{d+2} D_i^*) = \emptyset$, hence $P^*, C_1^*, \dots, C_{d+2}^*$ form a counterexample to Conjecture $(d+1, 1, d+2)$; this completes the proof in Case 1.

Case 2. $2 \leq t \leq d-2$. Let $P, C_1, \dots, C_{d-t+2}$ in E^{d-t+1} form a counterexample to Conjecture $(d-t+1, 1, d-t+2)$, given by Case 1 of the proof, and let $P^* = P \times I^{t-1}$, $C_i^* = C_i \times I^{t-1}$ for all $1 \leq i \leq d-t+2$, where I^{t-1} is the unit cube in E^{t-1} and $A \times B$ denotes the Cartesian product of A and B .

Let L be a t -flat in E^d that meets P^* ; $L \cap E^{d-t+1}$ is of dimension at least one, hence it (contains a line that) meets $\bigcup_{i=1}^{d-t+2} C_i^*$, therefore L meets $\bigcup_{i=1}^{d-t+2} C_i^*$.

If $D_1^*, \dots, D_{d-t+2}^*$ are polytopes in E^d , with $D_i^* \subseteq C_i^*$ for all $1 \leq i \leq d-t+2$, then define $D_1, \dots, D_{d-t+2}$ by taking D_i as the image of D_i^* under the orthogonal projection of E^d onto $\text{aff}(P)$. Clearly $D_i \subseteq C_i$ for all

$1 \leq i \leq d - t + 2$. Therefore there is a line L in $\text{aff}(P) = E^{d-t+1}$ that meets P and does not meet $\bigcup_{i=1}^{d-t+2} D_i$, hence $L \times E^{t-1}$ is a t -flat in E^d that meets P^* and does not meet $\bigcup_{i=1}^{d-t+2} D_i^*$. Therefore $P^*, C_1^*, \dots, C_{d-t+2}^*$ form a counterexample to Conjecture $(d, t, d - t + 2)$.

This completes the proof of Theorem 2.

Summary. Hoffman's Conjecture (d, t, k) was stated for all $d \geq 1, 0 \leq t \leq d - 1$ and $k \geq 1$; the following table summarizes the complete solution to Conjecture (d, t, k) :

d	t	k	Conjecture (d, t, k)	by
≥ 1	0	≥ 1	true	Hoffman [4].
≥ 1	> 0	1	trivially true	
≥ 1	> 0	2	true	Hare-Smith [3].
≥ 2	$d - 1$	≥ 2	true	Zaks [7].
≥ 3	$d - 2$	≥ 4	false ²	Zaks [6].
≥ 3	$1 \leq t \leq d - 2$	$1 \leq k \leq d - t + 1$	true	Theorem 1, this paper.
≥ 3	$1 \leq t \leq d - 2$	$k \geq d - t + 2$	false	Theorem 2, this paper.

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²This case is included in the last case of the table.