THE BRAUER GROUP OF A CLOSED CATEGORY

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ABSTRACT. The Brauer group of a finitely bicomplete closed category is defined. This group gives known Brauer groups for the appropriate choices of the closed category. There is a Brauer group functor from the category of commutative monoids in the closed category to the category of Abelian groups.

For a finitely bicomplete closed category $\mathbf{V}$ [8, p. 180] we define the Brauer group of $\mathbf{V}$, $B(\mathbf{V})$. This paper depends heavily on the calculus of Morita contexts given in [7]. If $\mathbf{V}$ is the category of left-modules over a commutative ring R , we have $B(\mathbf{V})$ is the Brauer group of R as defined in [2]; if $\mathbf{V}$ is the category of sheaves of modules over a commutative ringed space $(X, \mathcal{U})$, we have $B(\mathbf{V})$ is the Brauer group of $(X, \mathcal{U})$ as defined in [1]; if $\mathbf{V}$ is the category of $\mathbb{Z}/2\mathbb{Z}$ graded R -modules M with R a commutative ring concentrated in degree zero and with symmetric homomorphism $c: M \otimes M' \rightarrow M' \otimes M$ defined by

$$c(m \otimes m') = (-1)^{\delta m \cdot \delta m'} (m' \otimes m)$$

for homogeneous elements $m \in M$ and $m' \in M'$ of respective degrees δm and $\delta m'$, then $B(\mathbf{V})$ is the Brauer-Wall group of R as defined in [4]. We give necessary conditions on a functor T from one closed category $\mathbf{V}$ to another $\mathbf{V}'$ to insure that T induces a homomorphism from $B(\mathbf{V})$ to $B(\mathbf{V}')$. As a corollary we see that B induces a functor from the category of commutative monoids of $\mathbf{V}$ [8, p. 167] to the category of Abelian groups.

We recall that there is a one-to-one correspondence between monoids Λ in $\mathbf{V}$ and one-object $\mathbf{V}$ -categories $[\Lambda]$. The opposite monoid Λ° corresponds to the $\mathbf{V}$ -category $[\Lambda]^\circ$ and the enveloping monoid Λ^e corresponds to the $\mathbf{V}$ -category $[\Lambda] \otimes [\Lambda]^\circ$. We will denote the functor category $\mathbf{V}^{[\Lambda]}$ by

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V^A . A monoid Λ is said to be *commutative* if the identity map from Λ to Λ^o is a map of monoids.

For any V -category A the *center* of A , $Z(A)$, is defined by

$$Z(A) = \int_A A(A, A),$$

i.e., the end of the V -functor $A(-, -): A^o \otimes A \rightarrow V$, if it exists [6, (1.6)]. As an example $Z(V) = I$. We note that if $V_0: V^o \otimes V \rightarrow \text{Sets}$ is the underlying hom functor of V then $V_0(I, Z(A))$ is the set of natural transformations from the identity functor of A to itself. This set is called the center of the set based category A_0 by Bass [4, p. 55]. If A is small then $Z(A)$ always exists. If $Z(A)$ exists then it has the unique structure of a commutative monoid such that the projection maps $p_A: Z(A) \rightarrow A(A, A)$ are maps of monoids for every object A in A .

Proposition 1. *Let C be a small V -category where V is a bicomplete closed category. Then*

- (i) $Z(V^C)$ exists and $Z(V^C) = Z(C)$;
- (ii) $Z(C) = V^{C \otimes C^o}(C(-, -), C(-, -))$.

Proof. This follows from the V -Yoneda isomorphism [6, (1.19)] and the universal properties of ends. $\square$

If Λ is a monoid then its *center*, $Z(\Lambda)$, is defined to be $Z([\Lambda])$, which by Proposition 1 (ii) is $V^{A^e}(\Lambda, \Lambda)$. Call Λ *central* if the unit map $I \rightarrow Z(\Lambda)$ is an isomorphism. We note that for the remainder of this paper we need only assume V is finitely bicomplete.

Let $\text{Mon}(V)$ be the monoidal category with objects the monoids of V and maps the monoid isomorphisms. Let M be the category of Morita contexts with tensor product $\square$ and unit $I' = ([I] \rightarrow [I] \otimes G \leftarrow [I])$ as defined in [7]. Since we will be interested only in Morita contexts $M = (C, T)$ with C a V -category with two objects 0 and 1, we will use the notation

$$M = (C(0, 0), C(1, 0), C(0, 1), C(1, 1))$$

to specify the Morita contexts. The notation for M as a cospan then becomes

$$([\dot{C}(1, 1)] \rightarrow C \leftarrow [C(0, 0)])$$

and $\Phi(M') = C(1, 0) \otimes_{C(1, 1)} -$. Define $\Psi: \text{Mon}(V) \rightarrow M$ by $\Psi(\Lambda) = (C_\Lambda, T_\Lambda)$ where C_Λ is the full subcategory of V^{A^e} with objects Λ and Λ^e , and

T_Λ is defined by $T(\Lambda) = 0$ and $T(\Lambda^e) = 1$. We have

$$\Psi(\Lambda) = (Z(\Lambda), \Lambda, V^{\Lambda^e}(\Lambda, \Lambda^e), (\Lambda^e)^\circ).$$

We note that Ψ is a map of (symmetric) monoidal categories, i.e., we have natural maps

$$\psi_{\Lambda_1 \Lambda_2}: \Psi(\Lambda_1) \square \Psi(\Lambda_2) \rightarrow \Psi(\Lambda_1 \otimes \Lambda_2)$$

and a map $\psi_0: I' \rightarrow \Psi(I)$ satisfying diagrams MF 1–MF 4 in [5, pp. 473, 513]. In this case ψ_0 is the identity and if Λ_i is a V -atom in V^{Λ^e} for $i = 1, 2$, i.e., $V^{\Lambda^e}(\Lambda_i, -)$ is V -cocontinuous, then $\psi_{\Lambda_1 \Lambda_2}$ is an isomorphism, i.e., Ψ restricted to the category of such monoids is *strict*.

We call a monoid Λ an *Azumaya monoid* if Λ is central, Λ is a V -atom in V^{Λ^e} and the functor $[I] \rightarrow V^{\Lambda^e}$ induced by Λ is V -dense [6, after (1.17)]. In general if Ω_0 and Ω_1 are monoids and B is a left Ω_0 , right Ω_1 bimodule, i.e., B induces a functor $[\Omega_1]^\circ \otimes [\Omega_0] \rightarrow V$, then B is said to be Ω_0 - Ω_1 -*prodense* if B is a V -atom in V^{Ω_0} and the functor $B^\#: [\Omega_1]^\circ \rightarrow V^{\Omega_0}$ induced by B is V -dense and V -full and faithful. This is equivalent to B being a Ω_0 - Ω_1 -progenerator as defined in [3] if V is a category of left modules over a commutative ring. Thus Λ is an Azumaya monoid if and only if Λ is Λ^e - I -prodense.

Lemma 2. *If B is a left Ω_0 , right Ω_1 -bimodule the following statements are equivalent:*

- (a) B is Ω_0 - Ω_1 -prodense.
- (b) There is a Morita equivalence $M = (\Omega_0, B, D, \Omega_1)$.
- (c) B is Ω_1° - Ω_0° -prodense.
- (d) $V^{\Omega_0}(B, \Omega_0)$ is Ω_0° - Ω_1° -prodense and $V^{\Omega_0}(B, -): V^{\Omega_0} \rightarrow V^{\Omega_1}$ is

V -cocontinuous.

Proof. (a) $\Rightarrow$ (b) since $B^\#$ is full and faithful, $\Omega_1 = V^{\Omega_0}(B, B)$, and we get the Morita context in (b) by taking $D = V^{\Omega_0}(B, \Omega_0)$ and using the composition maps in V^{Ω_0} . Since B is a V -atom in V^{Ω_0} we have an isomorphism

$$V^{\Omega_0}(B, \Omega_0) \otimes_{\Omega_0} B \rightarrow V^{\Omega_0}(B, B) = \Omega_1$$

which implies M has a right $*$ -inverse. The density of $B^\#$ gives us that M is a Morita equivalence by [7, Theorem 4]. (b) $\Rightarrow$ (a) follows from [7, Theorem 4]. (c) $\Leftrightarrow$ (b) follows from (a) $\Leftrightarrow$ (b) by observing that M is a Morita equivalence if and only if $(M^\circ)^t = (\Omega_1^\circ, B, D, \Omega_0^\circ)$ is one. (b) $\Rightarrow$ (d) follows

from (b) $\Rightarrow$ (a) applied to $M^\circ = (\Omega_0^\circ, D, B, \Omega_1^\circ)$, since $D \simeq V^{\Omega_0}(B, \Omega_0)$ by [7, Theorem 4 (iii)] applied to M . (d) $\Rightarrow$ (a) follows from (a) $\Rightarrow$ (d) applied to $V^{\Omega_0}(B, \Omega_0)$ if we show $B \simeq V^{\Omega_0}_0(V^{\Omega_0}(B, \Omega_0), \Omega_0^\circ)$. This isomorphism follows from the fact that

$$V^{\Omega_0}(B, -) \quad \text{and} \quad V^{\Omega_0}(V^{\Omega_0}_0(V^{\Omega_0}(B, \Omega_0), \Omega_0^\circ), -)$$

are cocontinuous. $\square$

Let $I(V)$ be the monoidal subcategory of V with the same objects as V but with maps the isomorphisms of V . Define the map of monoidal categories $\theta: I(V) \rightarrow M$ by $\theta(X) = (D_X, T_X)$, where D_X is the full subcategory of V with objects X and I and T_X is defined by $T_X(X) = 0$ and $T_X(I) = 1$. When θ is restricted to the V -atoms of V it is a strict map. Note that $\theta(X) = (V(X, X), X, X^*, I)$ where $X^* = V(X, I)$.

Theorem 3. *For any object X of V the following statements are equivalent:*

- (a) *The functor $[V(X, X)] \rightarrow V$ induced by X is V -dense and X is a V -atom.*
- (b) *X is $V(X, X)$ - I -prodense.*
- (c) *$\theta(X)$ is a Morita equivalence.*
- (c') *$X \otimes: V \rightarrow V^{V(X, X)}$ is an equivalence of categories.*
- (d) *$\theta(X^*)$ is a Morita equivalence, $V(X, X) \simeq V(X^*, X^*)^\circ$, and X is a V -atom.*

Proof. Since (a) is equivalent to the statement that X is I - $V(X, X)^\circ$ -prodense, Lemma 2 implies the equivalence of (a), (b), (c) and the statement that X^* is I° - $V(X, X)^\circ$ -prodense and X is a V -atom in V . Note that if X^* is I° - $V(X, X)^\circ$ -prodense, then $(X^*)^\# : [V(X, X)]^\circ \rightarrow V$ is full and faithful and we get $V(X, X) \simeq V(X^*, X^*)^\circ$. The equivalence of (c) and (c') follows from [7, Corollary 5] applied to $\theta(X)^t$, since $\Phi(\theta(X)^t) = X \otimes -$. $\square$

Theorem 4. *For a monoid Λ the following statements are equivalent:*

- (a) *Λ is an Azumaya monoid.*
- (b) *$\Psi(\Lambda)$ is a Morita equivalence and Λ is central.*
- (b') *$\Lambda \otimes: V \rightarrow V^{\Lambda^e}$ is an equivalence of categories.*
- (c) *Λ is a V -atom in V , the functor $[\Lambda^e] \rightarrow V$ induced by Λ is dense and the canonical map $\Lambda^e \rightarrow V(\Lambda, \Lambda)$ is an isomorphism.*

Proof. The equivalence of (b) and (b') follows from [7, Corollary 5] since $\Phi(\Psi(\Lambda)^t) = \Lambda \otimes -$. The rest of the theorem follows from Lemma 2. $\square$

Theorem 5. (i) If Λ_1 and Λ_2 are Azumaya monoids then so is $\Lambda_1 \otimes \Lambda_2$.

(ii) If X_i is $V(X_i, X_i)$ - I -prodense, $i = 1, 2$, then $X_1 \otimes X_2$ is $V(X_1 \otimes X_2, X_1 \otimes X_2)$ - I -prodense, $V(X_1, X_1)$ is an Azumaya monoid, and $X_1 \otimes X_1^* \simeq V(X_1, X_1)$.

Proof. If Λ_1 and Λ_2 are Azumaya then $\Psi(\Lambda_1 \otimes \Lambda_2) \simeq \Psi(\Lambda_1) \square \Psi(\Lambda_2)$. Thus, by Theorem 4 and [7, Theorem 3] $\Lambda_1 \otimes \Lambda_2$ is Azumaya. If X_i is $V(X_i, X_i)$ - I -prodense, then X_i is a V -atom, $i = 1, 2$, and $\theta(X_1 \otimes X_2) \simeq \theta(X_1) \square \theta(X_2)$. Thus by Theorem 3 and [7, Theorem 3] $X_1 \otimes X_2$ is $V(X_1 \otimes X_2, X_1 \otimes X_2)$ - I -prodense. Since X_1^* is $V(X_1^*, X_1^*)$ - I -prodense, we obtain

$$\theta(X_1 \otimes X_1^*) \simeq \theta(X_1) \square \theta(X_1^*), \quad X_1 \otimes X_1^* \simeq V(X_1, X_1),$$

$$V(X_1, X_1)^\circ \simeq V(X_1^*, X_1^*).$$

This implies $V(X_1, X_1)$ is $V(X_1, X_1)^\circ$ - I -prodense and thus an Azumaya monoid. $\square$

We let $A(V)$ be the class of Azumaya monoids in V and $A_0(V)$ be the class of monoids $V(B, B)$ in V where B is $V(B, B)$ - I -prodense. By Theorem 5, $A(V)$ and $A_0(V)$ are closed under $\otimes$, and $A_0(V)$ is a subclass of $A(V)$. Define an equivalence relation on $A(V)$ by $\Lambda_1 \sim \Lambda_2$ if $V^{\Lambda_1} \simeq V^{\Lambda_2}$. Let the Brauer group of V , $B(V)$, be the equivalence classes under this equivalence relation. Theorems 3, 4 and 5 imply that $B(V)$ is a group (here we allow groups to have a class of objects) with identity element $\text{cl}(I) = \text{cl}(\Omega)$ for all Ω in $A_0(V)$, multiplication defined by $\text{cl}(\Lambda_1) \cdot \text{cl}(\Lambda_2) = \text{cl}(\Lambda_1 \otimes \Lambda_2)$ and inverse $\text{cl}(\Lambda)^{-1} = \text{cl}(\Lambda^\circ)$. We note that if V is cartesian closed then the unit object I is the only V -atom; hence $B(V)$ is trivial by Theorem 4.

We now consider maps of monoidal categories which induce maps of Brauer groups. First we need an alternate description of the equivalence relation defined above.

Proposition 6. For Λ_1 and Λ_2 Azumaya monoids, $\Lambda_1 \sim \Lambda_2$ if and only if there exist Ω_1 and Ω_2 in $A_0(V)$ such that $\Lambda_1 \otimes \Omega_1 \simeq \Lambda_2 \otimes \Omega_2$.

Proof. The necessity is clear since $V^\Omega \simeq V$ for every Ω in $A_0(V)$ by Theorem 3. For the sufficiency suppose $V^{\Lambda_1} \simeq V^{\Lambda_2}$. Then we have $V \simeq V^{\Lambda_1 \otimes \Lambda_1^\circ} \simeq V^{\Lambda_2 \otimes \Lambda_1^\circ}$, which implies by [7, Corollary 5] that there is a Morita

equivalence $(I, B, D, \Lambda_2 \otimes \Lambda_1^\circ)$ with B and D in $\mathbf{V}$, i.e., by Lemma 2, B is $I\text{-}\Lambda_2 \otimes \Lambda_1^\circ$ -prodense. This implies $\Lambda_2^\circ \otimes \Lambda_1 \simeq \mathbf{V}(B, B)$ and $\mathbf{V}(B, B)$ is in $A_0(\mathbf{V})$. Tensoring with Λ_2 gives the desired result since $\Lambda_2 \otimes \Lambda_2^\circ \simeq \mathbf{V}(\Lambda_2, \Lambda_2)$ is in $A_0(\mathbf{V})$. $\square$

Theorem 7. *Let $\mathbf{V}$ and $\mathbf{V}'$ be bicomplete closed categories and $T: \mathbf{V} \rightarrow \mathbf{V}'$ be a strict map of monoidal categories such that*

- (i) *T preserves atoms and coequalizers;*
- (ii) *There exists an isomorphism $K_B: T(\mathbf{V}(B, I)) \rightarrow \mathbf{V}'(TB, TI)$, for every atom B in $\mathbf{V}$, which is natural in B .*

Then T induces a group homomorphism $B(T): B(\mathbf{V}) \rightarrow B(\mathbf{V}')$.

Proof. Since T is strict we need only show T maps $A(\mathbf{V})$ into $A(\mathbf{V}')$ and $A_0(\mathbf{V})$ into $A_0(\mathbf{V}')$. But T is strict implies $T(\Lambda^e) \simeq T(\Lambda)^e$. For B a $\mathbf{V}$ -atom in $\mathbf{V}$, the isomorphism K_B induces natural isomorphisms

$$K_{B,A}: T(\mathbf{V}(B, A)) \rightarrow \mathbf{V}'(TB, TA)$$

for every A in $\mathbf{V}$ by the observation that for a $\mathbf{V}$ -atom B , $\mathbf{V}(B, I) \otimes A \simeq \mathbf{V}(B, A)$. Thus if Λ is Azumaya $T(\Lambda)^e \simeq \mathbf{V}(T\Lambda, T\Lambda)$. If B is $\mathbf{V}(B, B)$ - I -prodense then B is an atom and hence TB is one. To show TB is $\mathbf{V}'(TB, TB)$ - I' -prodense, it suffices to show that the map

$$f_{TB}: \mathbf{V}'(TB, TI) \otimes_{\mathbf{V}'(TB, TB)} TB \rightarrow I'$$

is an isomorphism. This follows from the facts that the corresponding map for B is an isomorphism and

$$\mathbf{V}'(TB, TI) \otimes_{\mathbf{V}'(TB, TB)} TB \simeq T(\mathbf{V}(B, I) \otimes_{\mathbf{V}(B, B)} B)$$

which holds since T preserves coequalizers and is strict and we have the isomorphisms K_{BI} and K_{BB} . Thus T maps $A_0(\mathbf{V})$ into $A_0(\mathbf{V}')$ and by using the Theorem 4 (c) characterization of Azumaya monoids we see T maps $A(\mathbf{V})$ into $A(\mathbf{V}')$. $\square$

Corollary 8. *If R and S are commutative monoids in $\mathbf{V}$ and $t: R \rightarrow S$ is a map of monoids, then there is a canonical group homomorphism $B(t): B(\mathbf{V}^R) \rightarrow B(\mathbf{V}^S)$ which sends $\text{cl}(\Lambda)$ to $\text{cl}(S \otimes_R \Lambda)$.*

Proof. We take $T = S \otimes_R -: \mathbf{V}^R \rightarrow \mathbf{V}^S$, the left adjoint of $\mathbf{V}^t: \mathbf{V}^S \rightarrow \mathbf{V}^R$, and observe that T satisfies the properties of Theorem 7. $\square$

Thus, B induces a functor from the category of commutative monoids in $\mathbf{V}$ to the category of Abelian groups.

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