A CHARACTERIZATION OF QUASICIRCLES

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ABSTRACT. In this paper the following result is established: a Jordan curve Γ in the extended plane $\bar{\mathbb{C}}$ is a quasicircle if and only if there is a K , $1 \leq K < \infty$, such that, given ordered triples z_1, z_2, z_3 and w_1, w_2, w_3 of points on Γ , there exists a K -quasiconformal mapping h of $\bar{\mathbb{C}}$ onto itself with $h(\Gamma) = \Gamma$ and $h(z_j) = w_j$ for $j = 1, 2, 3$.

Notation and terminology. For $1 \leq K < \infty$ we denote by Q_K the family of all K -quasiconformal mappings of the extended complex plane $\bar{\mathbb{C}}$ onto itself. A Jordan curve Γ in $\bar{\mathbb{C}}$ is said to be a *quasicircle* if, for some K , there is a map f in Q_K such that $f(\Gamma)$ is a euclidean circle. We say that Q_K is *triply-transitive* on a set A in $\bar{\mathbb{C}}$ if, given ordered triples z_1, z_2, z_3 and w_1, w_2, w_3 of points in A , there is a map g in Q_K such that $g(A) = A$ and $g(z_j) = w_j$ for $j = 1, 2, 3$.

Suppose that Γ is a Jordan curve in $\bar{\mathbb{C}}$ and that Q_K is triply-transitive on Γ . If b belongs to Q_{K_1} , it is clear that $Q_{K_1^2 K}$ is triply-transitive on $\Gamma_1 = b(\Gamma)$. Since Q_1 is precisely the class of Moeblus transformations of $\bar{\mathbb{C}}$ onto itself and since Q_1 is triply-transitive on circles and straight lines, it follows from the preceding remark that, given a quasicircle Γ in $\bar{\mathbb{C}}$, there exists a K such that Q_K is triply-transitive on Γ . It is the purpose of this note to show that this property characterizes quasicircles. The motivation for this result is found in the following observation.

Remark. If Q_1 is triply-transitive on the Jordan curve Γ in $\bar{\mathbb{C}}$, then Γ is a circle or straight line. Assuming, as we clearly may, that the points $0, 1$ and ∞ belong to Γ , we demonstrate that $\Gamma = \Gamma_0 = \mathbb{R} \cup \{\infty\}$. Let J be the finite subarc of Γ with endpoints 0 and 1 . We show that $J = J_0$, the closed unit interval on the real axis. Assuming this to be the case, let g

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be the unique Moebius transformation satisfying $g(0) = 1$, $g(1) = \infty$, $g(\infty) = 0$. Necessarily $g(\Gamma) = \Gamma$ and, of course, $g(\Gamma_0) = \Gamma_0$. Then

$$\Gamma = J \cup g(J) \cup g^2(J) = J_0 \cup g(J_0) \cup g^2(J_0) = \Gamma_0.$$

If $J \neq J_0$, we may assume there is a $b > 0$ such that J meets $H = \{z | \operatorname{Im}(z) > b\}$. Let E be any component of $J \cap H$. Then E is an (open) arc with endpoints $z_1 = a_1 + ib$ and $z_2 = a_2 + ib$, labeled so that $a_1 < a_2$. Let b be the unique Moebius transformation such that $b(z_1) = z_2$, $b(z_2) = \infty$, $b(\infty) = z_1$. Necessarily $b(\Gamma) = \Gamma$. On the other hand, $b(H) = H$ and, in particular, $b(E)$ and $b^2(E)$ are contained in H . Clearly

$$E \cup b(E) \cup b^2(E) = \Gamma - \{z_1, z_2, \infty\}.$$

Since 0 belongs to Γ and since $0 \neq z_1, z_2$, we infer that 0 is a point of H , an obvious contradiction. We conclude that $J = J_0$ and $\Gamma = \Gamma_0$, as asserted.

We now state our principal result.

Theorem. *A Jordan curve Γ in $\overline{\mathbb{C}}$ is a quasicircle if and only if there is a K such that $\mathcal{Q}_K$ is triply-transitive on Γ .*

Proof. We need only to establish the sufficiency. Let z_0 be an arbitrary point of Γ . We will find an open subarc J of Γ containing z_0 and a constant C with the following property: if x_1, x_2, x_3 are distinct points of J and if $\{x_1, x_3\}$ separates x_2 from the endpoints of J , then $q(x_1, x_2) \leq Cq(x_1, x_3)$, where q is the chordal metric in $\overline{\mathbb{C}}$. It will then follow from a theorem of Ahlfors that Γ is a quasicircle. (See [2] or [3, Theorem 8.7].)

In determining J and C we may assume that ∞ is a point of Γ and that $z_0 \neq \infty$. Otherwise we may choose a chordal isometry b belonging to $\mathcal{Q}_1$ such that $b(w_0) = \infty$, where $w_0 \neq z_0$ is a point of Γ . Then $\mathcal{Q}_K$ is triply-transitive on $\Gamma_1 = b(\Gamma)$, $z_1 = b(z_0) \neq \infty$ is a point of Γ_1 and Γ_1 contains $b(w_0) = \infty$. If J_1 and C satisfy the above requirements at z_1 , then $J = b^{-1}(J_1)$ and C satisfy them at z_0 .

Assuming that Γ passes through ∞ and that $z_0 \neq \infty$, choose any bounded, open subarc J of Γ containing z_0 . Denote the endpoints of J by y_1 and y_3 and set $M = \sup_J |z|$. We will find a constant C_1 such that, for x_1, x_2, x_3 on J in the order described above, $|x_1 - x_2| \leq C_1 |x_1 - x_3|$. Consequently, for such a triple of points, $q(x_1, x_2) \leq Cq(x_1, x_3)$, where $C = C_1(1 + M^2)$.

Fix x_1, x_2, x_3 on J so that $\{x_1, x_3\}$ separates x_2 from $\{y_1, y_3\}$.

There is a g belonging to Q_K such that $g(\Gamma) = \Gamma$, $g(x_1) = y_1$, $g(x_3) = y_3$, $g(\infty) = \infty$. It follows that $y_2 = g(x_2)$ is a point of J . Write

$$\alpha = \arcsin \frac{|y_1 - y_3|}{|y_1 - y_2| + |y_2 - y_3|}, \quad \beta = \arcsin \frac{|x_1 - x_3|}{|x_1 - x_2| + |x_2 - x_3|}.$$

By a distortion theorem of Agard and Gehring [1]

$$\sin \frac{\beta}{2} \geq \phi_K \left(\sin \frac{\alpha}{2} \right),$$

where $\phi_K: [0, 1] \rightarrow [0, 1]$ is strictly increasing and continuous, with $\phi_K(0) = 0$, $\phi_K(1) = 1$. Now

$$\frac{|y_1 - y_3|}{|y_1 - y_2| + |y_2 - y_3|} \geq \frac{|y_1 - y_3|}{4M} = \delta > 0,$$

from which we infer

$$\sin \frac{\beta}{2} \geq \phi_K \left(\sin \frac{\delta}{2} \right) = \gamma > 0.$$

We conclude that

$$\frac{|x_1 - x_3|}{|x_1 - x_2| + |x_2 - x_3|} \geq C_2 > 0,$$

with $C_2 = \sin(2 \arcsin \gamma)$. In particular, $|x_1 - x_2| \leq C_1 |x_1 - x_3|$, where $C_1 = C_2^{-1}$.

We have determined J and $C = C_1(1 + M^2)$ with the desired property and the proof of the Theorem is complete.

Remarks. The question immediately arises whether the double-transitivity or even simple transitivity of some Q_K on Γ is sufficient to imply that Γ is a quasicircle. We conjecture that double-transitivity is, in fact, sufficient, although we have thus far been unable to establish this. The difficulties seem to arise from the normalization $f(\infty) = \infty$ in the Agard-Gehring result. The question of simple transitivity appears to be more delicate. We are not at all convinced that Γ will necessarily be a quasicircle in this case.

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