ZONOIDS WHOSE POLARS ARE ZONOIDS

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ABSTRACT. It is shown that Euclidean n -space contains nonellipsoidal, centrally symmetric convex bodies which, as well as their polars, are zonoids (i.e., can be approximated by finite sums of segments). This disproves a conjecture of E. D. Bolker.

A *zonoid* in Euclidean n -space $\mathbf{R}^n$ is a convex body which can be approximated (with respect to the Hausdorff metric) by finite vector sums of line-segments. There are several equivalent definitions of zonoids (e.g., as ranges of nonatomic vector measures) for which the reader is referred to Bolker [1]. In [1, p. 340], and again in [2], Bolker conjectured that the only zonoids in $\mathbf{R}^n$ ($n \geq 3$), whose polar bodies (taken with respect to the centre) are also zonoids, are the ellipsoids. Equivalently this would mean (compare [1, p. 340]) that the Euclidean space $\mathbf{R}^n$ ($n \geq 3$) is the only n -dimensional Banach space which is isometric to a subspace of L^1 and to a quotient space of L^∞ . However, Bolker's conjecture is untrue as we shall now show.

Let

$$S = \{x \in \mathbf{R}^n: \langle x, x \rangle = 1\}$$

denote the unit sphere of $\mathbf{R}^n$, and let

$$B = \{x \in \mathbf{R}^n: \langle x, x \rangle \leq 1\}$$

be its unit ball ($\langle \cdot, \cdot \rangle$ is the scalar product of $\mathbf{R}^n$). For a real function F on S for which it makes sense we denote the sup-norm by $\|F\|$ and the L^2 -norm with respect to Lebesgue measure ω on S by $\|F\|_L$. For F of class C^{2r} let

$$\|F\|_r = a_n \|F\|_L + b_n \|\Delta^r F\|_L,$$

where Δ denotes the Laplace operator on S and where a_n, b_n are positive

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constants which will be defined later. The support function H_K (restricted to S) of a convex body $K \subset \mathbf{R}^n$ is defined by

$$H_K(y) = \sup\{\langle x, y \rangle : x \in K\}$$

for $y \in S$. By $\mathfrak{E}_n$ we denote the set of all convex bodies in $\mathbf{R}^n$ which are centrally symmetric with respect to the origin and whose support function is of class C^{2r} , where $2r = n + 2$ or $n + 3$, according to whether n is even or odd. On $\mathfrak{E}_n$ we define a metric d by $d(K, L) = \|H_K - H_L\|_r$.

Proposition. *In $(\mathfrak{E}_n, d)$, a whole neighborhood of B consists of zonoids.*

Proof. We shall use the fact that the convex body K is a zonoid if there exists a positive Borel measure μ on S such that $H_K(y) = \int |\langle x, y \rangle| d\mu(x)$, $y \in S$ (all integrations are over S). For $K \in \mathfrak{E}_n$ there exists a uniquely determined even, continuous function h_K on S for which

$$(1) \quad H_K(y) = \int |\langle x, y \rangle| h_K(x) d\omega(x), \quad y \in S;$$

see [4]. A slight extension of the argument used in [4] will yield a bound for $\|h_K\|$ in terms of $\|H_K\|_r$.

For real continuous functions F, G on S we define the inner product by

$$(F, G) = \int FG d\omega.$$

For $m = 0, 1, 2, \dots$ let $\{Y_{mj} : j = 1, \dots, N(n, m)\}$ be an orthonormal basis of the space of spherical surface harmonics of degree m on S . Using the addition theorem for spherical harmonics (Müller [3, p. 10]), the Cauchy-Schwarz inequality, and the equality (22) of [3, p. 15], we arrive at

$$(2) \quad \left| \sum_{j=1}^{N(n, m)} (F, Y_{mj}) Y_{mj}(x) \right| \leq \left[\frac{N(n, m)}{\omega_n} \right]^{1/2} \|F\|_L, \quad x \in S,$$

where $\omega_n = \omega(S)$.

If $K \in \mathfrak{E}_n$, the function h_K which satisfies (1) is given by

$$(3) \quad h_K(x) = \sum_{m=0; 2|m}^{\infty} \lambda_m^{-1} \sum_{j=1}^{N(n, m)} (H_K, Y_{mj}) Y_{mj}(x), \quad x \in S,$$

where

$$\lambda_m = \frac{(-1)^{\frac{1}{2}(m-2)} \pi^{\frac{1}{2}(n-1)} \Gamma(m-1)}{2^{m-2} \Gamma(\frac{1}{2}m) \Gamma(\frac{1}{2}(m+n+1))},$$

see [4]. As noticed in [4], repeated application of Green's formula leads to

$$(4) \quad (H_K, Y_{mj}) = [-1/m(m+n-2)]^r (\Delta^r H_K, Y_{mj}),$$

$$j = 1, \dots, N(n, m), \quad m = 1, 2, \dots$$

From (3), (4), and (2) it follows that

$$|b_K(x)| \leq |\lambda_0^{-1}| \omega_n^{-\frac{1}{2}} \|H_K\|_L + \sum_{m=2; 2|m}^{\infty} |\lambda_m^{-1}| m^{-2r} \left[\frac{N(n, m)}{\omega_n} \right]^{\frac{1}{2}} \|\Delta^r H_K\|_L,$$

$$x \in S.$$

If we now assume that a_n, b_n have been defined by

$$a_n = |\lambda_0^{-1}| \omega_n^{-\frac{1}{2}}, \quad b_n = \sum_{m=2; 2|m}^{\infty} |\lambda_m^{-1}| m^{-2r} \left[\frac{N(n, m)}{\omega_n} \right]^{\frac{1}{2}}$$

(see [4] for the convergence of the latter series), then we have $\|b_K\| \leq \|H_K\|_r$.

Clearly the function b_B is a positive constant. Hence if $K \in \mathfrak{E}_n$ and $d(K, B) \leq b_B$, then

$$\|b_K - b_B\| \leq \|H_K - H_B\|_r = d(K, B) \leq b_B,$$

which implies that b_K is nonnegative and, therefore, that K is a zonoid.

This proves the Proposition.

It is now easy to construct nonellipsoidal zonoids in $\mathbf{R}^n$ whose polars are also zonoids. Choose an even real function f on S of class C^∞ which does not vanish identically, and put $x_\lambda(u) = [1 + \lambda f(u)]u$ for $u \in S$ and $\lambda > 0$. For λ sufficiently small, the map $x_\lambda: S \rightarrow \mathbf{R}^n$ defines the boundary of a centrally symmetric convex body K_λ . Let $e_\lambda(u)$ denote the exterior unit normal vector to $x_\lambda(S)$ at $x_\lambda(u)$. For λ sufficiently small, the map $e_\lambda: S \rightarrow S$ is a diffeomorphism; let e_λ^{-1} denote the inverse diffeomorphism. Then $H_\lambda(v) = \langle x_\lambda(e_\lambda^{-1}(v)), v \rangle$, $v \in S$, is the support function of the body K_λ , while the support function of its polar K_λ^* is simply given by $H_\lambda^*(u) = [1 + \lambda f(u)]^{-1}$, $u \in S$. It is now a matter of standard calculations to show that $\lambda > 0$ can be chosen so small that $K_\lambda, K_\lambda^* \in \mathfrak{E}_n$ and that $\|H_\lambda - H_B\|_r \leq b_B$ and $\|H_\lambda^* - H_B\|_r \leq b_B$, which implies that both K_λ and K_λ^* are zonoids.

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