HYPONORMAL OPERATORS WITH INFINITE ESSENTIAL SPECTRUM

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ABSTRACT. It is shown that if T is an essentially hyponormal operator (i.e., the image of $T^*T - TT^*$ in the Calkin algebra is a positive element) in $\mathfrak{L}(\mathcal{H})$, and if the left essential spectrum of T is infinite, then $R(\delta_T)^- + \{T^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$, where $R(\delta_T)^-$ denotes the norm closure of the range of derivation induced by T , and $\{T^*\}'$ denotes the commutant of T^* .

Introduction. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\delta_A(X) = AX - XA$ be an inner derivation on the algebra $\mathfrak{L}(\mathcal{H})$ of all bounded, linear operators on $\mathcal{H}$. The range of δ_A will be denoted by $R(\delta_A)$, the closure of $R(\delta_A)$ in the norm topology will be denoted by $R(\delta_A)^-$, and the commutant of A will be denoted by $\{A\}'$. Throughout this paper, the term operator will always mean a bounded operator in $\mathfrak{L}(\mathcal{H})$, $\Pi_0(A)$ will denote the point spectrum (i.e., the set of eigenvalues) of A , and $\mathbb{C}$ will denote the complex field. In what follows, the spectrum of an operator A will be denoted by $\sigma(A)$, and the left essential spectrum will be denoted by $E_l(A)$: we note that $E_l(A)$ is the set of all complex numbers λ such that the image of $A - \lambda$ in the Calkin algebra is not left invertible. If $\mathcal{C}_2$ denotes the Hilbert space of Hilbert-Schmidt operators in $\mathfrak{L}(\mathcal{H})$, then it is known [8, p. 38] that every operator A in $\mathfrak{L}(\mathcal{H})$ induces a direct sum decomposition $\mathcal{C}_2 = R(\delta_A)^- \oplus \{A^*\}' \cap \mathcal{C}_2$, where $R(\delta_A)^-$ denotes the closure of $\{AX - XA : X \in \mathcal{C}_2\}$ in the Hilbert-Schmidt norm. The fact that $R(\delta_A)^-$ is the orthogonal complement of $\{A^*\}' \cap \mathcal{C}_2$ in $\mathcal{C}_2$ for every operator A in $\mathfrak{L}(\mathcal{H})$ indicates that the norm closure of $R(\delta_A)^- + \{A^*\}'$ is a natural subspace of $\mathfrak{L}(\mathcal{H})$, which is associated with each operator A in $\mathfrak{L}(\mathcal{H})$.

Recently J. Anderson [1] has shown that if N is a normal operator, then

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$R(\delta_N)^- + \{N^*\}'$ is norm dense in $\mathfrak{L}(\mathcal{H})$ if and only if $\sigma(N)$ is finite. However, $R(\delta_{N+K})^-$ is not, in general, complemented by $\{N^* + K^*\}'$ in $\mathfrak{L}(\mathcal{H})$, even if the spectrum of N is finite: a simple counterexample is given at the end of Corollary 4.

The purpose of this paper is to continue the investigation of when $R(\delta_A)^- + \{A^*\}'$ is norm dense in $\mathfrak{L}(\mathcal{H})$. The main result of this paper shows that if D is a diagonalable normal operator in $\mathfrak{L}(\mathcal{H})$ of uniform infinite multiplicity (i.e., each eigenvalue of D has infinite multiplicity), and if $\Pi_0(D)$ is infinite, then for every compact operator K , $R(\delta_{D+K})^- + \{D^* + K^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$. We also show, as a corollary, that if T is an essentially hyponormal operator (i.e., the image of $T^*T - TT^*$ in the Calkin algebra is a positive element), and if $E_I(T)$ is infinite, then $R(\delta_T)^- + \{T^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$.

We begin with our main theorem.

Theorem 1. *If D is a diagonalable normal operator in $\mathfrak{L}(\mathcal{H})$ of uniform infinite multiplicity, and if $\Pi_0(D)$ is infinite, then for every compact operator K , $R(\delta_{D+K})^- + \{D^* + K^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$.*

Proof. Let $\{\lambda_n: n = 1, 2, \dots\}$ be a sequence of distinct eigenvalues of D , which converges to a fixed point λ_0 , and let K be a fixed compact operator in $\mathfrak{L}(\mathcal{H})$. By assumption, the subspace $\mathfrak{E}_n = \{x \in \mathcal{H}: Dx = \lambda_n x\}$ is infinite dimensional for every positive integer n . Since $\|Kx_n\| \rightarrow 0$ for every infinite sequence of orthonormal vectors $\{x_n\}$ in $\mathcal{H}$, and since $\mathfrak{E}_n$ is infinite dimensional for every n , there exists an eigenvector e_n in $\mathfrak{E}_n$ such that $De_n = \lambda_n e_n$, $\|e_n\| = 1$, and $\|K^*e_n\| \leq |\lambda_n - \lambda_{n+1}|/2n$ for each positive integer n .

Similarly, there exists an eigenvector f_n in $\mathfrak{E}_{n+1}$ such that $Df_n = \lambda_{n+1}f_n$, $\|f_n\| = 1$, and $\|Kf_n\| \leq |\lambda_n - \lambda_{n+1}|/2n$ for every n . By the way e_n 's and f_n 's are defined, it is obvious that $(\|K^*e_n\| + \|Kf_n\|)/|\lambda_n - \lambda_{n+1}| \leq 1/n$ for every positive integer n .

Hence

$$(1) \quad \lim_{n \rightarrow \infty} \frac{\|K^*e_n\| + \|Kf_n\|}{|\lambda_n - \lambda_{n+1}|} = 0.$$

Let U be a partial isometry so that $Ue_n = f_n$ for every positive integer n , and let $U = 0$ on the orthogonal complement of $M = \bigvee_{n=1}^{\infty} e_n$: for our purpose, it is sufficient to let U be any bounded operator on $M^\perp$. We will now show that

$$\alpha = \|(D + K)X - X(D + K) + S - U\| \geq 1$$

for each fixed X in $\mathfrak{L}(\mathcal{H})$ and S in $\{D^* + K^*\}'$.

A simple calculation shows that

$$\begin{aligned}\beta_n &= (((D + K)X - X(D + K))e_n, f_n) = ((DX - XD)e_n, f_n) + ((KX - XK)e_n, f_n) \\ &= (\lambda_{n+1} - \lambda_n)(Xe_n, f_n) + ((KX - XK)e_n, f_n).\end{aligned}$$

Since $|\lambda_{n+1} - \lambda_n| \rightarrow 0$ and $KX - XK$ is a compact operator,

$$(2) \quad \lim_{n \rightarrow \infty} |\beta_n| = 0.$$

Using the fact that S commutes with $D^* + K^*$, we have

$$\begin{aligned}(S(D^* + K^*)e_n, f_n) &= \bar{\lambda}_n(Se_n, f_n) + (SK^*e_n, f_n) = ((D^* + K^*)Se_n, f_n) \\ &= (Se_n, \lambda_{n+1}f_n) + (K^*Se_n, f_n) = \bar{\lambda}_{n+1}(Se_n, f_n) + (K^*Se_n, f_n).\end{aligned}$$

Therefore,

$$(\bar{\lambda}_n - \bar{\lambda}_{n+1})(Se_n, f_n) = ((K^*S - SK^*)e_n, f_n)$$

and

$$|(Se_n, f_n)| \leq \frac{|((K^*S - SK^*)e_n, f_n)|}{|\lambda_n - \lambda_{n+1}|} \leq \|S\| \frac{\|K^*e_n\| + \|Kf_n\|}{|\lambda_n - \lambda_{n+1}|}.$$

By (1), it follows that $\lim_{n \rightarrow \infty} |(Se_n, f_n)| = 0$. Since $\alpha \geq |\beta_n + (Se_n, f_n) - (Ue_n, f_n)| = |\beta_n + (Se_n, f_n) - 1|$ for all positive integers n , and since $\lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} (Se_n, f_n) = 0$, it is obvious that $\alpha = \|(D + K)X - X(D + K) + S - U\| \geq 1$ for every X in $\mathfrak{L}(\mathcal{H})$ and S in $\{D^* + T^*\}'$.

Therefore, $R(\delta_{D+K})^- + \{D^* + K^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$ for every compact operator K .

If, in the proof of Theorem 1, we considered an operator S in $\{D + K\}'$, instead of S in $\{D^* + K^*\}'$, the results of (1) and (2) would still be valid. Therefore, under the assumption of Theorem 1, $R(\delta_{D+K})^- + \{D + K\}'$ is also not dense in $\mathfrak{L}(\mathcal{H})$, in the norm topology, for every compact operator K .

In the following, we derive several corollaries of our main theorem.

Corollary 2. *Let $T = D \oplus T_1$ be an operator in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$. If D is a diagonal normal operator in $\mathfrak{L}(\mathcal{H})$ of uniform infinite multiplicity, and if $\Pi_0(D)$ is infinite, then for every compact operator K in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$, $R(\delta_{T+K})^- + \{T^* + K^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$.*

Proof. Let K be a fixed compact operator in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$, and let $\{\lambda_n : n = 1, 2, \dots\}$ be a sequence of distinct eigenvalues of D which converges to a point λ_0 . By Theorem 1, there exists a sequence of orthonormal

eigenvectors $\{e_n, f_n: n = 1, 2, \dots\}$ such that

$$De_n = \lambda_n e_n, \quad Df_n = \lambda_{n+1} f_n, \quad \text{and} \quad \frac{\|K^* e_n\| + \|K f_n\|}{|\lambda_n - \lambda_{n+1}|} \leq \frac{1}{n}$$

for every positive integer n . If we let U be the partial isometry that maps each e_n to f_n , $n = 1, 2, \dots$, and let $U = 0$ on the orthogonal complement of $\bigvee_{n=1}^{\infty} e_n$, then the rest of the proof follows exactly as in the case of Theorem 1.

The following definition and notations are needed in the sequel. Let π denote the canonical quotient map from $\mathfrak{L}(\mathcal{H})$ onto the Calkin algebra $\mathfrak{L}(\mathcal{H})/\mathcal{K}$, where $\mathcal{K}$ denotes the closed ideal of all compact operators in $\mathfrak{L}(\mathcal{H})$.

Definition. A complex number λ is a reducing essential eigenvalue of T in $\mathfrak{L}(\mathcal{H})$ if there exists a projection P in $\mathfrak{L}(\mathcal{H})$ such that $\pi(P) \neq 0$ and $\pi(T - \lambda)\pi(P) = \pi(T^* - \bar{\lambda})\pi(P) = 0$. The set of all reducing essential eigenvalues of an operator T will be denoted by $R_e(T)$.

Corollary 3. Let T be an operator in $\mathfrak{L}(\mathcal{H})$. If $R_e(T)$ is infinite, then $R(\delta_T)^- + \{T^*\}'$ is not norm dense.

Proof. If $R_e(T)$ is infinite, then, by Theorem 4.5 in [6, p. 570], T is unitarily equivalent to an operator $(D \oplus T_1) + K$ in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$, where D is a diagonal normal operator in $\mathfrak{L}(\mathcal{H})$ of uniform infinite multiplicity, $\Pi_0(D)$ is infinite, $T_1 \in \mathfrak{L}(\mathcal{H})$, and K is a compact operator in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$. Therefore, by Corollary 2, $R(\delta_T)^- + \{T^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$.

Corollary 4. If T is an essentially hyponormal operator (i.e., $\pi(T^*)\pi(T) \geq \pi(T)\pi(T^*)$), and if $E_I(T)$ is infinite, then $R(\delta_T)^- + \{T^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$.

Proof. If T is an essentially hyponormal operator, it is known that $R_e(T) = E_I(T)$ (see [6, p. 567]). Therefore, by Corollary 2, $R(\delta_T)^- + \{T^*\}'$ is not norm dense in $\mathfrak{L}(\mathcal{H})$.

Some open questions. If $D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, then $\sigma(D)$ is obviously finite, and hence, by [1, Theorem 2.4], $R(\delta_D)^- + \{D^*\}' = \mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$. On the other hand, if

$$D + K = \begin{pmatrix} I & 0 \\ 0 & K_{22} \end{pmatrix},$$

where K_{22} is a diagonalable compact operator with distinct entries on its diagonal, then $R(\delta_{D+K})^- + \{D^* + K^*\}'$ is easily shown to be not dense in $\mathfrak{L}(\mathcal{H} \oplus \mathcal{H})$ in the norm topology. Therefore, the following question is of some interest.

Question 1. For which nonnormal operator T , is it true that $R(\delta_T)^- + \{T^*\}'$ is norm dense in $\mathcal{L}(\mathcal{H})$?

For an arbitrary operator A in $\mathcal{L}(\mathcal{H})$, it is in general not known whether $R(\delta_A) \cap \{A^*\}' = \{0\}$.

Question 2. If T is a nonnormal operator, what are the necessary and sufficient conditions for $R(\delta_T) \cap \{T^*\}' = \{0\}$?

By a theorem of Klejnecke and Shirokov [4], [7], $R(\delta_A) \cap \{A\}'$ consists entirely of quasinilpotent operators for every operator A in $\mathcal{L}(\mathcal{H})$. Therefore, if $T \in R(\delta_A) \cap \{A, A^*\}'$, then it is easy to show that $T^*T \in R(\delta_A) \cap \{A, A^*\}'$, and the quasinilpotence of T^*T implies that $T = 0$. Therefore, $R(\delta_A) \cap \{A, A^*\}' = \{0\}$ for every operator A in $\mathcal{L}(\mathcal{H})$. Because every compact operator K in $R(\delta_A)^- \cap \{A\}'$ is shown to be quasinilpotent by Theorem 2 in [3], $R(\delta_A)^- \cap \{A, A^*\}'$ does not contain any nonzero compact operator for every operator A in $\mathcal{L}(\mathcal{H})$. However, there exists an operator A such that $R(\delta_A)^- \cap \{A, A^*\}' \neq \{0\}$. In particular, if $I \in R(\delta_A)^-$, then $R(\delta_A)^- \supseteq \{A\}'$: the existence of such an operator A was established by J. Anderson in [2].

Question 3. If T is a nonnormal operator, what are the necessary and sufficient conditions for $R(\delta_T)^- \cap \{T, T^*\}' \neq \{0\}$?

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