HOMOMORPHIC IMAGES OF σ -COMPLETE BOOLEAN ALGEBRAS

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ABSTRACT. It is a well-known theorem of R. S. Pierce that, for every infinite cardinal α , $\alpha^{\aleph_0} = \alpha$ if and only if there is a complete Boolean algebra B s.t. $\text{card } B = \alpha$ (see [3, Theorem 25.4]). Recently, Comfort and Hager proved [1] that, for every infinite σ -complete Boolean algebra B , $(\text{card } B)^{\aleph_0} = \text{card } B$. We extend this result to the class of homomorphic images of σ -complete algebras, following closely Comfort's and Hager's proof. As a corollary, an improvement of Shelah's theorem on the cardinality of ultraproducts of finite sets [2] is derived (Theorem 2).¹

We denote the finite operations on a Boolean algebra A by $+$, $\cdot$, and $-$, the infinite operations by Σ and Π . If $a \in A$ and $\alpha > 0$, $A|a$ is the algebra $\{x \in A \mid x \leq a\}$. We write $\text{card } a$ for $\text{card } A|a$ (if $a = 0$, $\text{card } a = 1$). A sequence $\langle a_n \mid n \in \omega \rangle$ in A is disjointed if $a_n \neq 0$ and $a_n \cdot a_m = 0$ for $n, m \in \omega$, $n \neq m$. For every sequence $\langle A_n \mid n \in \omega \rangle$ of Boolean algebras, $\prod_{n \in \omega} A_n$ is the product algebra.

Theorem 1. *Let B be a σ -complete Boolean algebra, p an epimorphism from B onto A and $\alpha = \text{card } A \geq \aleph_0$. Then $\alpha^{\aleph_0} = \alpha$.*

Proof. We first prove three lemmas.

(a) *Suppose $x, y \in B$ s.t. $x \leq y$, put $a = p(x)$ and $b = p(y)$. If $c \in A$ s.t. $a \leq c \leq b$, there is some $z \in B$ s.t. $x \leq z \leq y$ and $p(z) = c$. If $d \in A$ s.t. $a \cdot d = 0$, there is some $t \in B$ s.t. $x \cdot t = 0$ and $p(t) = d$.*

Proof. Choose $z', t' \in B$ s.t. $p(z') = c$, $p(t') = d$; put $z = z' \cdot y + x$, $t = t' \cdot -x$.

(b) *Suppose I is an ideal of A s.t. $\text{card } I = \text{card } A = \alpha \geq 2^{\aleph_0}$; every countable subset of I has an upper bound in I and for every $a \in I$, $(\text{card } a)^{\aleph_0}$ equals $2^{\aleph_0}$ or $\text{card } a$. Then $\alpha^{\aleph_0} = \alpha$.*

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¹ This answers a question of M. Ziegler whether Theorem 1 allows to generalize Shelah's result to arbitrary reduced products of finite sets; the author's original proof only established Shelah's theorem.

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Proof.

$$\begin{aligned} \alpha^{\aleph_0} &= \text{card} \{f \mid f: \aleph_0 \rightarrow I\} = \text{card} \{f \mid \text{for some } a \in I, f: \aleph_0 \rightarrow A|a\} \\ &\leq \text{card } I \cdot \max \left(2^{\aleph_0}, \sup_{a \in I} \text{card } a \right) \leq \alpha \cdot \max(2^{\aleph_0}, \alpha) = \alpha. \end{aligned}$$

(c) Suppose $\langle a_n \mid n \in \omega \rangle$ is a disjointed sequence in A . Then $\text{card} \prod_{n \in \omega} (A|a_n) \leq \alpha$.

Proof. We shall construct a one-one mapping ϕ from $P = \prod_{n \in \omega} (A|a_n)$ to A . Let f be a function (not necessarily a homomorphism) from A to B s.t. $p \circ f = \text{id}_A$. By (a) there is a disjointed sequence $\langle b_n \mid n \in \omega \rangle$ in B s.t. $p(b_n) = a_n$ for $n \in \omega$. Define, for $\langle x_n \mid n \in \omega \rangle \in P$,

$$\phi(\langle x_n \mid n \in \omega \rangle) = p \left(\sum_{n \in \omega} f(x_n) \cdot b_n \right).$$

Since $\phi(\langle x_n \mid n \in \omega \rangle) \cdot a_m = x_m$ for every $m \in \omega$, ϕ is one-one.

Returning to the proof of the theorem, suppose $\alpha < \alpha^{\aleph_0}$ and that α is the least cardinal providing a counterexample. By (c), $2^{\aleph_0} \leq \alpha$. Let $I = \{a \in A \mid \text{card } a < \alpha\}$. I is a proper ideal of A . If $a \in I$ and $0 < a$, $A|a$ is a homomorphic image of A , hence of B and thus, by minimality of α , $(\text{card } a)^{\aleph_0}$ equals $2^{\aleph_0}$ or $\text{card } a$.

The algebra $C = A/I$ is finite, for otherwise, by (a), there would exist a disjointed sequence $\langle a_n \mid n \in \omega \rangle$ in $A \setminus I$, and by (c), $\alpha^{\aleph_0} \leq \alpha$. Thus $\text{card } I = \alpha$. If every countable subset of I has an upper bound in I , we have reached a contradiction. Thus, assume the contrary.

There is some $k \in \omega \setminus \{0\}$ s.t. $\text{card } C = 2^k$ and some disjointed sequence $\langle a_j \mid j < k \rangle$ in $A \setminus I$. We may assume that $\sum_{j < k} a_j = 1$. There is some $j < k$ s.t. not every countable subset of $I \cap A|a_j$ has an upper bound in $I \cap A|a_j$. Put $A' = A|a_j$ and $I' = I \cap A'$. Then A' is a homomorphic image of A and hence of B , $\text{card } A' = \alpha$, I' is a prime ideal in A' , $\text{card } I' = \alpha$, and for every $a \in I'$ s.t. $0 < a$, $(\text{card } A'|a)^{\aleph_0}$ equals $2^{\aleph_0}$ or $\text{card } A'|a$. We may and do assume that $A = A'$ and $I = I'$, i.e. that I is a prime ideal of A . By the above assumption, there is a disjointed sequence $\langle c_n \mid n \in \omega \rangle$ in I s.t. $\{c_n \mid n \in \omega\}$ has no upper bound in I . Therefore, the ideal $K = \{k \in A \mid k \cdot c_n = 0 \text{ for every } n \in \omega\}$ is a subset of I . By (c), $\text{card} \prod_{n \in \omega} (A|c_n) \leq \alpha$. We actually have $\text{card} \prod_{n \in \omega} (A|c_n) < \alpha$, for otherwise,

$$\alpha^{\aleph_0} = \left(\text{card} \prod_{n \in \omega} (A|c_n) \right)^{\aleph_0} = \prod_{n \in \omega} (\text{card } A|c_n)^{\aleph_0} = \prod_{n \in \omega} (\text{card } A|c_n) = \alpha.$$

The range of the homomorphism $f: A \rightarrow \prod_{n \in \omega} (A|c_n)$, defined by $f(a) = \langle a \cdot c_n | n \in \omega \rangle$ for $a \in A$, thus has cardinality less than α , and its kernel K has cardinality α . We shall show that every countable subset of K has an upper bound in K , which contradicts (b). Suppose $\langle k_n | n \in \omega \rangle$ is a sequence in K . Define, by (a), inductively κ_n and δ_n in B s.t.

$$p(\kappa_0) = k_0,$$

$$\begin{aligned} p(\delta_0) &= -c_0, & \kappa_0 &\leq \delta_0, \\ &\dots & & \\ p(\kappa_{n+1}) &= k_0 + \dots + k_{n+1}, & \kappa_n &\leq \kappa_{n+1} \leq \delta_n, \\ p(\delta_{n+1}) &= -(c_0 + \dots + c_{n+1}), & \kappa_{n+1} &\leq \delta_{n+1} \leq \delta_n. \end{aligned}$$

Then $d = p(\prod_{n \in \omega} \delta_n)$ is an upper bound of $\{k_n | n \in \omega\}$ in K .

Corollary. *Suppose A is an infinite homomorphic image of some complete Boolean algebra. Then $(\text{card } A)^{\aleph_0} = \text{card } A$.*

Remark. Suppose A is a Boolean algebra satisfying

- (1) if $M, N \subseteq A$ are countable and $m \cdot n = 0$ for every $m \in M, n \in N$, then there exists $a \in A$ s.t. $m \leq a, a \cdot n = 0$ for every $m \in M, n \in N$.

Then $(\text{card } A)^{\aleph_0} = \text{card } A$, provided $\text{card } A \geq \aleph_0$. In fact, property (1) is all that is needed in the proof of Theorem 1. And it is clear that homomorphic images of σ -complete Boolean algebras and $\aleph_1$ -saturated Boolean algebras satisfy (1). This shows a close connection between Shelah's method and ours.

We shall use two trivial facts about reduced products. Let I be a nonvoid set and F a filter on I .

Lemma 1. *Let X_i be an arbitrary nonvoid set for every $i \in I$. Put $J = \{i \in I | \text{card } X_i = 1\}$, $K = I \setminus J$. If $J \in F$, $\text{card } \prod_{i \in I} X_i / F = 1$. If $J \notin F$, $G = \{M \cap K | M \in F\}$ is a filter on K and there is a canonical isomorphism from $\prod_{i \in I} X_i / F$ onto $\prod_{i \in K} X_i / G$.*

Lemma 2. *Let X_i, Z_i be arbitrary sets for every $i \in I$. Then there is a canonical isomorphism from $\prod_{i \in I} (X_i \times Z_i) / F$ onto $\prod_{i \in I} X_i / F \times \prod_{i \in I} Z_i / F$.*

Theorem 2. *Suppose $I \neq \emptyset$, F is a filter on I , $n_i \in \omega$ for $i \in I$, and $\alpha = \text{card } \prod_{i \in I} n_i / F \geq \aleph_0$. Then $\alpha^{\aleph_0} = \alpha$.*

Proof. For every $x = \langle x_i | i \in I \rangle \in \omega^I$, put $c(x) = \text{card } \prod_{i \in I} x_i / F$. We

first show:

(*) Suppose $x, y \in \omega^I$, $2 \leq x_i \leq y_i \leq 2x_i$ for every $i \in I$ and $c(x) \geq \aleph_0$. Then $c(y) = c(x)$.

We clearly may assume $y_i = 2x_i$ for every $i \in I$. But then we have, by Lemma 2, $c(y) = \text{card } 2^I / F \cdot c(x) = c(x)$.

To prove the theorem, let $n = \langle n_i | i \in I \rangle \in \omega^I$ be given s.t. $c(n) \geq \aleph_0$. By Lemma 1, we may assume $2 \leq n_i$ for every $i \in I$. Define, for $i \in I$,

$$y_i = 2^{k_i+1}, \quad \text{where } k_i \in \omega \text{ and } 2^{k_i} \leq n_i < 2^{k_i+1}.$$

Since $n_i < y_i \leq 2n_i$, we have, by (*), $\alpha = c(n) = c(y)$. Every y_i may be considered as a finite (and hence, complete) Boolean algebra. Thus $\prod_{i \in I} y_i$ carries, in a canonical way, the structure of a complete Boolean algebra and $A = \prod_{i \in I} y_i / F$ the structure of a homomorphic image of a complete Boolean algebra. By the Corollary, $\alpha = c(y) = \text{card } A = \aleph_0$.

Remarks. 1. It is easily seen that, if F is an ultrafilter on I and $y_i = 2^{k_i}$ for every $i \in I$ s.t. $\text{card } \prod_{i \in I} y_i / F \geq \aleph_0$, then the algebra $A = \prod_{i \in I} y_i / F$ includes a disjointed sequence $\langle a_n | n \in \omega \rangle$ s.t. $\text{card } \dot{a}_n = \text{card } A$ for every $n \in \omega$. This establishes, together with the lemma (c) in Theorem 1, a short proof of Shelah's theorem.

2. The Corollary not only implies Theorem 2, but is, in fact, equivalent to it: let B be complete, $p: B \rightarrow A$ an epimorphism and $\text{card } A \geq \aleph_0$. Since B is an injective Boolean algebra, there is some set I and some epimorphism q from the power set of I onto B . Put $F = \{x \subseteq I | p(q(x)) = 1\}$. Then A is isomorphic to the reduced product $2^I / F$.

3. The question naturally arises whether there is also a simple way to prove Theorem 1 from the Corollary, i.e. whether every σ -complete Boolean algebra is the homomorphic image of some complete Boolean algebra. This problem seems to be unsolved. The only fact I know is

Proposition (CH). Assume A is a σ -complete Boolean algebra s.t. $\text{card } A \leq 2^{\aleph_0}$. Then there is a complete Boolean algebra B and a homomorphism p from B onto A .

Proof. If A is finite, A is complete. So, let A be infinite. Thus, $\text{card } A = 2^{\aleph_0} = \aleph_1$. Put $A = \{a_\alpha | \alpha < \aleph_1\}$. Let F be the free Boolean algebra on $\aleph_1$ free generators and let B be F^* , the MacNeille completion of F . Since $\text{card } B = 2^{\aleph_0} = \aleph_1$, put $B = \{b_\alpha | \alpha < \aleph_1\}$. We construct a sequence $\langle p_\alpha | \alpha < \aleph_1 \rangle$ of homomorphisms from subalgebras of B to A s.t. $\alpha \leq \beta < \aleph_1$ implies $p_\alpha \subseteq p_\beta$, $p = \bigcup_{\alpha < \aleph_1} p_\alpha$ is a homomorphism from B onto A and each

p_α has countable domain. Let p_0 be the homomorphism from the two-element subalgebra of B onto the two-element subalgebra of A . If λ is a countable limit ordinal, put $p_\lambda = \bigcup_{\alpha < \lambda} p_\alpha$. Now suppose $\alpha < \aleph_1$ and p_α has been constructed s.t. p_α has countable domain.

Case 1. α is even. Let b be the element of B with the least subscript s.t. $b \notin \text{dom } p_\alpha$. Since $\text{dom } p_\alpha$ is countable and A is σ -complete, an examination of the proof of Sikorski's extension theorem shows that p_α may be extended to the subalgebra of B generated by b and $\text{dom } p_\alpha$.

Case 2. α is odd. Let a be the element of A with least subscript s.t. $a \notin \text{rng } p_\alpha$. Since $\text{dom } p_\alpha$ is countable, there is some $b \in B$ s.t. b is independent of $\text{dom } p_\alpha$. Hence p_α may be extended to the subalgebra of B generated by b and $\text{dom } p_\alpha$, putting $p_{\alpha+1}(b) = a$.

REFERENCES

1. W. W. Comfort and A. W. Hager, *Cardinality of $\aleph_1$ -complete Boolean algebras*, Pacific J. Math. 40 (1972), 541–545. MR 46 #7112.
2. S. Shelah, *On the cardinality of ultraproducts of finite sets*, J. Symbolic Logic 35 (1970), 83–84.
3. R. Sikorski, *Boolean algebras*, 2nd ed., Springer-Verlag, Berlin and New York, 1964.

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