SPECTRA OF COMPACT COMPOSITION OPERATORS

JAMES G. CAUGHRAN AND HOWARD J. SCHWARTZ

ABSTRACT. Let ϕ be holomorphic and map the open unit disk into itself, and let $C_\phi: f \mapsto f \circ \phi$ be the composition operator on H^2 generated by ϕ . If C_ϕ is a compact operator then (1) $\phi(z_0) = z_0$ for some $z_0 \in D$; (2) $\sigma(C_\phi) = \{\phi'(z_0)^n: n = 0, 1, 2, \dots\} \cup \{0\}$.

Let ϕ be holomorphic and nonconstant in the open unit disk D and suppose $|\phi(z)| < 1$ for $z \in D$. For f holomorphic in D , define the *composition operator* C_ϕ by $(C_\phi f)(z) = (f \circ \phi)(z) = f(\phi(z))$. Then C_ϕ is a bounded linear operator on the Hilbert space H^2 [6].

In this paper, we describe the spectrum of C_ϕ in the case in which C_ϕ is compact. (For a discussion of geometric conditions which guarantee compactness, see [7].) To describe the spectrum we prove a theorem which may be of independent interest: that if C_ϕ is compact, ϕ has a fixed point in D .

We are grateful to several mathematicians for conversations about this material, including James Deddens, Alan Lambert, Peter Rosenthal, and Joel Shapiro. The first author would like to express his appreciation to the National Research Council of Canada for partial support while this work was in progress.

The *reproducing kernel* for H^2 , $k_\zeta(z) = (1 - \bar{\zeta}z)^{-1}$, $\zeta \in D$, is characterized by the property that for $f \in H^2$, $(f, k_\zeta) = f(\zeta)$. The function k_ζ is itself an H^2 function, of norm $(1 - |\zeta|^2)^{-1/2}$. In terms of these functions, we can characterize the composition operators among operators on H^2 .

Theorem 1. *Let A be an operator on H^2 . Then A is a composition operator if and only if the image of every kernel function under A^* is a kernel function. If $\zeta \in D$, $C_\phi^* k_\zeta = k_{\phi(\zeta)}$.*

Proof. If $A = C_\phi$ is a composition operator, then

Presented to the Society, April 1, 1972 under the title *Spectra of composition operators on H^2* ; received by the editors December 26, 1973.

AMS (MOS) subject classifications (1970). Primary 30A78, 47A10; Secondary 30A31, 47B05.

Copyright © 1975, American Mathematical Society

$$(f, A^*k_\zeta) = (C_\phi f, k_\zeta) = f(\phi(\zeta)) = (f, k_{\phi(\zeta)}),$$

so $A^*k_\zeta = k_{\phi(\zeta)}$ for every $\zeta \in D$.

Conversely, if A^* takes each kernel function k_ζ into a kernel function $k_{\phi(\zeta)}$, then if $f \in H^2$,

$$(Af)(\zeta) = (Af, k_\zeta) = (f, A^*k_\zeta) = (f, k_{\phi(\zeta)}) = f(\phi(\zeta)).$$

Letting f be the identity function $f(z) = z$, we see ϕ is holomorphic and $A = C_\phi$.

Theorem 2. *If C_ϕ is a compact composition operator, then $\phi(z)$ has a fixed point in D .*

Proof. Since C_ϕ is compact, ϕ is not a rotation about a fixed point [7]. By the Wolff-Denjoy theorem [2], [8], there is a point $\alpha \in \bar{D}$ so that for every $z \in D$, the iterates $\phi_n(z)$ converge to α , where $\phi_0(z) = z$, $\phi_{n+1}(z) = \phi(\phi_n(z))$. If $|\alpha| < 1$, then $\phi(\alpha) = \alpha$.

Suppose, by way of contradiction, that $|\alpha| = 1$. Since $\lim_{n \rightarrow \infty} \phi_n(0) = \alpha$, for infinitely many n , $|\phi_{n+1}(0)| > |\phi_n(0)|$. Choose points $\zeta_n = \phi_n(0)$ from among these values of n . Let k_{ζ_n} be the kernel function for ζ_n . Then $C_\phi^* k_{\zeta_n} = k_{\zeta_{n+1}}$. Because C_ϕ^* is compact, the images of $k_{\zeta_n} / \|k_{\zeta_n}\|$ under C_ϕ^* have a subsequence $\{g_j\}$ which converges in norm. Since

$$\|g_j\| = \|k_{\zeta_{n+1}}\| / \|k_{\zeta_n}\| = (1 - |\phi_n(0)|^2)^{1/2} / (1 - |\phi_{n+1}(0)|^2)^{1/2} > 1,$$

the norm of the limit function g is at least 1.

On the other hand,

$$\begin{aligned} (g, g_j) &= (1 - |\zeta_n|^2)^{1/2} (g, k_{\zeta_{n+1}}) = \left(\frac{1 - |\zeta_n|^2}{1 - |\zeta_{n+1}|^2} \right)^{1/2} (1 - |\zeta_{n+1}|^2)^{1/2} g(\zeta_{n+1}) \\ &< \|C_\phi\| o(1) \end{aligned}$$

since $\lim_{|z| \rightarrow 1} (1 - |z|)^{1/2} f(z) = 0$ for $f \in H^2$ [3, Theorem 5.9]. So $\|g\|^2 = \lim (g, g_j) = 0$. This contradiction shows $|\alpha| < 1$.

Corollary. *If C_ϕ^N is compact for some $N > 0$, then $\phi(z)$ has a fixed point in D .*

Proof. Because $C_\phi^N = C_{\phi_N}$, for some $\alpha \in D$, $\phi_N(\alpha) = \alpha$, and thus $\phi_{kN}(\alpha) = \alpha$ for positive integers k . A periodic sequence converges only if constant, so $\phi(\alpha) = \alpha$.

Theorem 3. If C_ϕ^N is compact for some $N > 0$, then $\sigma(C_\phi) = \{\phi'(z_0)^n : n = 1, 2, \dots\} \cup \{0, 1\}$, where z_0 is the fixed point of ϕ .

Proof. By an operator similarity, we may assume the fixed point is at 0. For if τ is a linear fractional transformation of D onto D taking z_0 to 0, then

$$C_\tau C_\phi C_\tau^{-1} = C_{\tau \circ \phi \circ \tau^{-1}},$$

and $\tau \circ \phi \circ \tau^{-1}(0) = 0$. By the chain rule, $(\tau \circ \phi \circ \tau^{-1})'(0) = \phi'(z_0)$.

The matrix of C_ϕ^* with respect to the basis $\{z^k\}$ is upper triangular with diagonal entries $1, \overline{\phi'(0)}, \overline{\phi'(0)^2}, \overline{\phi'(0)^3}, \dots$ [1], so $\phi'(0)^n \in \sigma(C_\phi)$. Thus $\{\phi'(0)^n : n = 1, 2, \dots\} \cup \{0, 1\} \subseteq \sigma(C_\phi)$.

It remains to show that the only possible eigenvalues for C_ϕ are 1 and $\phi'(0)^n$ for some n . This is essentially due to Koenig [4]. If $f \circ \phi = \lambda f$, then $f(0) = \lambda f(0)$, so if $\lambda \neq 1$, $f(0) = 0$. Write $f(z) = z^m g(z)$, where g is holomorphic in D and $g(0) \neq 0$. It follows that $\phi(z)^m g(\phi(z)) = \lambda z^m g(z)$, and thus $(\phi(z)/z)^m g(\phi(z)) = \lambda g(z)$. The value of $\phi(z)/z$ at $z = 0$ is $\phi'(0)$, so $\lambda = \phi'(0)^m$.

Since the nonzero spectrum of a power-compact operator consists of eigenvalues, the theorem is proved.

REFERENCES

1. J. G. Caughran, *Polynomial approximation and spectral properties of composition operators on H^2* , Indiana Univ. Math. J. 21 (1971/72), 81–84. MR 44 #4213.
2. M. A. Denjoy, *Sur l'iteration des fonctions analytiques*, C. R. Acad. Sci. Paris 182 (1926), 255–257.
3. P. L. Duren, *Theory of H^p spaces*, Pure and Appl. Math., vol. 38, Academic Press, New York, 1970. MR 42 #3552.
4. M. Koenig, *Recherches sur les intégrals de certaines équations fonctionnelles*, Ann. Sci. École Norm. Sup. 3 (1884), 1–112.
5. E. A. Nordgren, *Composition operators*, Canad. J. Math. 20 (1968), 442–449. MR 36 #6961.
6. J. V. Ryff, *Subordinate H^p functions*, Duke Math. J. 33 (1966), 347–354. MR 33 #289.
7. J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on H^2* , Indiana Univ. Math. J. 23 (1973), 471–496.

8. J. Wolff, *Sur l'iteration des fonctions...*, C. R. Acad. Sci. Paris 182 (1926), 42–43, 200–201.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUERTO RICO, MAYAGUEZ,
PUERTO RICO 00708

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY, LONG BEACH,
CALIFORNIA 90804 (Current address of H. J. Schwartz)

Current address (J. G. Caughran): 809 Dynes Road, Burlington, Ontario L7N 2P7,
Canada