ON THE SUM OF DARBOUX FUNCTIONS

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ABSTRACT. The main purpose is to indicate how badly the sum of a continuous function with a Darboux function can fail to be a Darboux function.

1. Introduction. In recent years a number of articles have dealt with questions concerning the possible outcomes of adding two real functions with the Darboux property (i.e. the intermediate value property). For example, Sierpiński [8] (see also Fast [4]) showed that in the absence of further conditions on the functions, *every* function is the sum of two such (Darboux) functions. If one imposes certain additional conditions on the functions, this imposes restrictions on the possible sums, but the only functions f such that $f + d$ is a Darboux function for each Darboux function d , are the constant functions. A survey of results involving sums of Darboux functions can be found in §7 of [1]. Some more recent work on the subject can be found in [2], [5], and [6].

Since the sum of a nonconstant continuous function and a Darboux function might fail to be a Darboux function, it is natural to ask just how badly it can fail. Not every function is such a sum. In fact, such a sum must be the uniform limit of a sequence of Darboux functions, and this imposes severe restrictions which are analyzed in [3]. On the other hand, Švarc [9] has recently obtained two theorems that show that to certain continuous functions f and sets M there correspond Darboux functions d such that $f + d$ is unbounded in every interval yet the range of $f + d$ misses M . The main purpose of the present paper is to obtain in §3 a result, Theorem 1, which extends Švarc's two theorems by placing almost no restrictions on the set M except the obviously necessary condition that M have a dense complement and almost no restriction on the continuous function. In §4 we obtain a

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general theorem related to the types of sums under consideration. This theorem also contains the two theorems of Švarc.

2. Preliminaries. In the sequel we shall be concerned entirely with real functions defined on the real line R . We shall let $\mathcal{D}$ denote the class of Darboux functions, $\mathcal{D}^*$ denote the class of functions which take on every real value in every interval, and $\mathcal{D}^{**}$ denote the class of functions which take on every real value c times in every interval, where c denotes the cardinality of the continuum. It is clear that $\mathcal{D}^{**} \subset \mathcal{D}^* \subset \mathcal{D}$. If A is a set of real numbers we shall denote its cardinality by $|A|$, its Lebesgue measure (if it is measurable) by λA . If B is a planar set, we shall denote its x -projection by $\text{dom } B$ and its y -projection by $\text{rng } B$. We shall denote the set of natural numbers by ω_0 . Finally, where convenient, we shall not distinguish a function from its graph.

3. The main result. We are now ready to obtain the result mentioned in the Introduction. Actually, we obtain a bit more, as we assume only that f have the Darboux property rather than that f be continuous.

Theorem 1. *Let f be a Darboux function which is constant on no subinterval of R and let M be a set of real numbers whose complement is dense. Then, for each countable dense subset D of $R \setminus M$ there exists a function $d \in \mathcal{D}^*$ such that the range of $f + d$ is D .*

Proof. Let D be a countable dense subset of $R \setminus M$ and let $B = \{(x, r - f(x)): r \in D\}$. We shall define a function $d \in \mathcal{D}^*$ such that $d \subseteq B$. This will guarantee that $f(x) + d(x) \in D$ for all x .

Let $\mathcal{H}$ consist of all horizontal lines and $\mathcal{V}$ consist of all vertical lines. If $z \in R^2$, then $H(z)$ or $V(z)$ will denote that member of $\mathcal{H}$ or $\mathcal{V}$ containing z . It is clear that for each $V \in \mathcal{V}$, $V \cap B$ is $\aleph_0$ -dense in V .

For $H \in \mathcal{H}$ and J any open subinterval of R there exist x_1 and x_2 in J for which $f(x_1) \neq f(x_2)$, since f is not constant on any subinterval. Now D is dense in R , so there exists an $r \in D$ such that $\lambda = \text{rng } H$ lies between $r - f(x_1)$ and $r - f(x_2)$. Since f is Darboux there exists an $x \in J$ for which $r - f(x) = \lambda$. Therefore, $H \cap B$ is dense in H .

For each planar set A , let $M(A) = \{B \cap H: \text{there exists } z \in A \text{ such that } V(z) \cap B \cap H \neq \emptyset\}$. Let $G_0(A) = M(A)$ and define

$$G_{n+1}(A) = M(G_n(A)) \quad \text{and} \quad G(A) = \bigcup_{n=0}^{\infty} G_n(A).$$

Let $\mathcal{N} = \{G(H): H \in \mathcal{H}\}$. In order to define the desired function d , we need

the following facts:

- (1) $|\text{rng } N| = \aleph_0$ for all $N \in \mathcal{N}$;
- (2) $\bigcup \mathcal{N} = B$;
- (3) N is dense in $H \cap N$ for all $H \in \mathcal{H}$ and $N \in \mathcal{N}$;
- (4) if $N_1, N_2 \in \mathcal{N}$ and $\text{dom } N_1 \cap \text{dom } N_2 \neq \emptyset$, then $N_1 = N_2$.

Parts (2), (3) and (4) can be verified without difficulty. For the proof of (1) it suffices to show that $|\text{rng } G_{n+1}(H)| = \aleph_0$ whenever $|\text{rng } G_n(H)| = \aleph_0$. To show this let $r \in D$ and $K \in \mathcal{H}$ and put $A(K, r) = \{y: \text{there exists } x \text{ such that } (x, r - f(x)) \in K \text{ and } (x, y) \in B\}$. The set $A(K, r)$ is easily seen to have cardinality $\aleph_0$. Putting $A(K) = \bigcup \{A(K, r): r \in D\}$ we have $|A(K)| = \aleph_0$ for each $K \in \mathcal{H}$. Then for $H \in \mathcal{H}$ and $|\text{rng } G_n(H)| = \aleph_0$ we have $\text{rng } G_{n+1}(H) = \bigcup \{A(K): \text{rng } K \subseteq \text{rng } G_n(H)\}$ so that $\text{rng } G_{n+1}(H)$ is a countable union of countable sets and hence $|\text{rng } G_{n+1}(H)| = \aleph_0$ finishing the proof of (1).

We can now define the function d . First we define for each $N \in \mathcal{N}$ a function $d_N \subseteq N$ such that d_N is dense in any nonempty $H \cap N$ for $H \in \mathcal{H}$. To do this let $\{O_n\}_{n=0}^\infty$ be an enumeration of all horizontal rational open intervals which intersect N . Pick $w_0 \in O_0 \cap N$ and $w_{n+1} \in O_{n+1} \cap N - \bigcup_{i=0}^n V(w_i)$. Put $d_N = \{w_i: i \in \omega_0\}$. Then d_N is easily seen to be a function dense in each nonempty $H \cap N$.

Next enumerate the countable family of uncountable sets of the form $\{(x, y): x \in J, y = r - f(x)\}$, where J is a rational open interval and $r \in D$, as $\{C_i\}_{i=0}^\infty$. By induction pick a sequence of points $\{e_i\}_{i=0}^\infty$ such that $e_0 \in C_0$ and $e_i \in C_i \setminus \bigcup \{N \in \mathcal{N}: \text{there exists } j < i \text{ with } e_j \in N\}$. This choice is possible because the range of C_i is an interval whereas the range of the other set is countable.

Define the function e by $e = \{e_i: i \in \omega_0\}$. Then $|e \cap N| \leq 1$ for any $N \in \mathcal{N}$ so that $|e \cap d_N| \leq 1$ for any $N \in \mathcal{N}$.

Now define a function d on R as follows:

$$d(x) = \begin{cases} e(x) & \text{if } x \in \text{dom } e, \\ d_N(x) & \text{if } x \in \text{dom } d_N \setminus \text{dom } e, \\ \text{any point in } V((x, 0)) \cap B & \text{if } x \notin \bigcup \{\text{dom } d_N: N \in \mathcal{N}\} \cup \text{dom } e. \end{cases}$$

It is clear that $d \subseteq B$. Since each horizontal open interval O hits some N and $O \cap d_N \setminus e$ is dense in O , it follows that d takes on each value infinitely often over each subinterval. Therefore, $d \in \mathcal{D}^*$.

For any open interval I and $r \in D$ the function e and hence d hits the set $\{(x, y): x \in I, y = r - f(x)\}$. Hence, $\text{rng}(f + d) \supset I = D$.

This finishes the proof.

Remark 1. It is easy to verify that we cannot drop the requirements that $R \setminus M$ be dense or that f be constant from our hypotheses without falsifying the conclusion in general. Švarc assumed in his two theorems that f and/or M meet certain additional conditions namely, that the level sets of f are countable and M is countable, or f is nowhere constant and M is nowhere dense. Theorem 1 now shows these assumptions are unnecessary.

It follows from Theorem 1 that to each continuous function f which is constant on no interval there corresponds a function $d \in \mathcal{D}^*$ such that the range of $f + d$ is countable. Thus, the graph of the sum of a continuous f and a $d \in \mathcal{D}^*$ can miss all but a countable number of horizontal lines. Now a function in $\mathcal{D}^*$ is already somewhat pathological because it takes on every value in every interval. But such functions can be well behaved in other respects, for example, there exist such functions in the second class of Baire. One might suppose that if f is sufficiently well behaved, then d can be chosen to be reasonably well behaved, but this does not seem to be true. Theorem 2 below shows in particular that if $R \setminus M$ is countable and f is absolutely continuous, then d cannot even be Lebesgue measurable.

Theorem 2. *If f is defined on R and absolutely continuous on an interval I , and if $d \in \mathcal{D}^*$ is measurable, then $f + d$ must have an uncountable range.*

Proof. We note first that since f is absolutely continuous, it satisfies Lusin's condition (N) and Banach's condition (T_1) . Suppose that the range of $f + d$ were a countable set. Without loss of generality we may suppose this set is infinite and enumerated as $\{m_k\}_{k=1}^\infty$. Let $E_k = \{x: d(x) + f(x) = m_k\}$. Then $I = \bigcup_{k=1}^\infty E_k$, and $E_k \cap E_l = \emptyset$ if $k \neq l$. Since d is measurable, each of the sets E is measurable. Now $d(E_k) = (m_k - f)(E_k)$ for each k , and this set is measurable because f satisfies Lusin's condition (N). It follows that

$$\lambda d(E_k) = \lambda f(E_k) \quad \text{for each } k.$$

Now let

$$A = \{x: f \text{ has no finite or infinite derivative at } x\},$$

$$B = \{x: f'(x) \text{ is finite}\},$$

and

$$C = \{x: f'(x) \text{ is infinite}\}.$$

Then (1) $\lambda f(A) = 0$ because f satisfies Banach's condition (T_1) (see Saks [7, p. 278]), (2) $\lambda f(B) \leq \int_B |f'| d\lambda$ (see [7, p. 272]), and (3) $\lambda f(C) = 0$ because

$\lambda C = 0$ and f satisfies Lusin's condition (N). Thus

$$\infty = \lambda d(I) \leq \sum_{k=1}^{\infty} \lambda d(E_k) = \sum_{k=1}^{\infty} \lambda f(E_k \cap B) \leq \sum_{k=1}^{\infty} \int_{E_k \cap B} |f'| d\lambda = \int_B |f'| d\lambda < \infty,$$

a contradiction. Thus the range of $d + f$ cannot be countable.

Remark 2. We have been unable to determine whether Theorem 2 remains valid if we weaken absolute continuity of f to continuity of f .

4. A general result. We now turn to the question of characterizing those functions f which, for a given set M , admit a $d \in \mathcal{D}^{**}$ such that the range of $f + d$ misses M . Theorem 3 below gives such a characterization, and we observe that Švarc's results follow readily from this theorem.

Theorem 3. *Let M be any proper subset of R and f be a real function on R . Then there exists a $d \in \mathcal{D}^{**}$ such that the range of $f + d$ misses M if and only if $|\{x \in I: \lambda + f(x) \notin M\}| = c$ for every $\lambda \in R$ and every interval I .*

Proof. ($\Leftarrow$) Suppose that $|\{x \in I: \lambda + f(x) \notin M\}| = c$ for each interval I and real number λ . Let $\mathcal{Q}$ be the set of all nonvoid open intervals in R . Well-order the set $(R \times \mathcal{Q}) \cup (\mathcal{Q} \times (R - M))$ as $\{z_\alpha\}_{\alpha < c}$. Note that $R \times \mathcal{Q}$ and $\mathcal{Q} \times R \setminus M$ are disjoint. We shall define the required function d through the use of transfinite induction. The product $R \times \mathcal{Q}$ will be used to guarantee that $d \in \mathcal{D}^{**}$ and the product $\mathcal{Q} \times (R \setminus M)$ will be used to guarantee that the range of $f + d$ over each interval equals $R \setminus M$. Suppose that a point $(x_\alpha, y_\alpha) \in R^2$ has been chosen for all $\alpha < \beta$. We wish to define (x_β, y_β) . We have two cases:

Case I. $z_\beta \in \mathcal{Q} \times (R \setminus M)$ and $z_\beta = (I, \lambda)$ where $I \in \mathcal{Q}$ and $\lambda \in R \setminus M$. Since $|\{x_\alpha: \alpha < \beta\}| < c$ we may pick $x_\beta \in I \setminus \{x_\alpha: \alpha < \beta\}$. Then put $y_\beta = \lambda - f(x_\beta)$.

Case II. $z_\beta \in R \times \mathcal{Q}$ and $z_\beta = (\lambda, I)$ where $I \in \mathcal{Q}$ and $\lambda \in R$. Since $|\{x \in I: \lambda + f(x) \notin M\}| = c$, we may pick $x_\beta \in I \setminus \{x_\alpha: \alpha < \beta\}$ so that $\lambda + f(x_\beta) \notin M$. Then put $y_\beta = \lambda$.

Our construction thus far defines a function d on the set $E = \{x_\alpha: \alpha < c\}$, namely $d(x_\alpha) = y_\alpha$. We now extend this function to all of R . Thus, if x is not in E , choose $d(x)$ in R so that $d(x) + f(x) \notin M$. This is possible because $M \neq R$.

We now show that the function d has the desired properties. First of all, for any nonvoid subinterval J and $\lambda \in R$ the cardinality of all open subintervals of J is c so that $|\{\beta: z_\beta = (\lambda, I) \text{ where } I \in \mathcal{Q} \text{ and } I \subseteq J\}| = c$. But for each β in this set $y_\beta = \lambda$ so that $|\{x \in J: d(x) = \lambda\}| = c$. Hence, $f \in \mathcal{D}^{**}$.

Secondly, for any $\lambda \in R \setminus M$ and any interval I we have $d(x_\beta) + f(x_\beta) = \lambda$ when $z_\beta = (I, \lambda)$ by Case I. Hence, $\text{rng}(f + d) \restriction I \supseteq R \setminus M$. On the other hand, we have $f(x_\beta) + d(x_\beta) = \lambda \in R \setminus M$ whenever $z_\beta \in \mathcal{U} \times R \setminus M$ and we have $d(x_\beta) + f(x_\beta) \notin M$ whenever $z_\beta \in R \times \mathcal{U}$. Moreover, for $x \notin \text{dom } E$, $d(x) + f(x) \notin M$. Hence, in any case $\text{rng}(f + d) \restriction I \subseteq R \setminus M$. Therefore $\text{rng}(f + d) \restriction I = R \setminus M$ for all subintervals I .

($\Rightarrow$) Suppose there exists a Darboux function d with the properties stated in the statement of the theorem.

Suppose there exists $\lambda \in R$ and an open interval I for which $\{x \in I: \lambda + f(x) \notin M\} < c$. Since $\{x \in I: d(x) = \lambda\} = c$ there exists $x \in I$ such that $d(x) = \lambda$ and $\lambda + f(x) \in M$. Therefore, $d(x) + f(x) \in M$, a contradiction. Hence, $\{x \in I: \lambda + f(x) \notin M\} = c$ for all $\lambda \in R$ and intervals I . This completes the proof of Theorem 3.

Theorem 3 does not hold if we replace $\mathcal{D}^{**}$ with $\mathcal{D}^*$. For example, suppose $M = \{0\}$ and $f(x) = 0$ if x is irrational and $f(x) = 1$ when x is rational. Let $d \in \mathcal{D}^*$ with $d(x) = 0$ when x is rational. Then $\text{rng}(f + d) \restriction I = R - \{0\}$ for each I . Moreover $\{x \in I: d(x) = \lambda\} \leq c$ and in fact $\{x \in I: d(x) = 0\} < c$. However, $\{x \in I: 0 + f(x) \notin \{0\}\} = \aleph_0$.

We have been unable to find a necessary and sufficient condition for the existence of a $d \in \mathcal{D}^*$ having the property that $\text{rng}(f + d) \restriction I \supseteq R \setminus M$ for each interval I .

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