PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 51, Number 2, September 1975

HEREDITARILY CLOSURE-PRESERVING COLLECTIONS AND METRIZATION

D. BURKE, R. ENGELKING AND D. LUTZER[1]

ABSTRACT. In this paper we present a generalization of the Nagata-Smirnov metrization theorem. We prove that a regular T_1-space is metrizable if and only if it has a base of open sets which is the union of countably many hereditarily closure-preserving subcollections. In addition, we investigate intersections of hereditarily closure-preserving collections of open sets.

A collection $\mathcal{K}$ of subsets of a space X is *closure-preserving* if $\mathrm{cl}(\bigcup\mathcal{L}) = \bigcup\{\mathrm{cl}(L) \mid L \in \mathcal{L}\}$ for any subcollection $\mathcal{L}$ of $\mathcal{K}$. A collection $\mathcal{H}$ of subsets of X is *hereditarily closure-preserving* (HCP) if, whenever a subset $K(H) \subset H$ is chosen for each $H \in \mathcal{H}$, the resulting collection $\mathcal{K} = \{K(H) \mid H \in \mathcal{H}\}$ is closure-preserving. Clearly, every locally finite collection is hereditarily closure-preserving. Examples show that closure preserving collections may fail to be HCP and that HCP collections may fail to be locally finite. A *σ-HCP collection* is one which can be written as a countable union of HCP subcollections.

The classical Nagata-Smirnov metrization theorem [4], [5] asserts that a regular space[2] is metrizable if and only if it has a σ-locally finite base. Regular spaces which have a σ-closure-preserving base were introduced by J. Ceder [1]. Ceder called these spaces "M_1-spaces" and gave examples which show that M_1-spaces need not be first-countable and that even when they are first-countable they need not be metrizable.

In this paper we consider regular spaces which have a σ-HCP base. Such spaces appear to lie between metrizable spaces and M_1-spaces. We will show that they coincide with metrizable spaces.

Lemma 1. *Let X be a T_1-space and suppose $p \in X$ has a neighborhood*

Received by the editors August 10, 1973 and, in revised form, June 14, 1974.
AMS (MOS) subject classifications (1970). Primary 54E35; Secondary 54D20.
Key words and phrases. Metrizability, Nagata-Smirnov theorem, hereditarily closure-preserving collection.
[1] Partially supported by NSF Grant GP 29401.
[2] We adopt the convention that regular spaces must be T_1.
Copyright © 1975, American Mathematical Society

base $\mathcal{B}$ *of cardinality* $\mathfrak{m}$. *Let* $\mathcal{H}$ *be an HCP collection of subsets of* X *and suppose that no member of* $\mathcal{H}$ *contains* p. *Then some neighborhood of* p *meets fewer than* $\mathfrak{m}$ *members of* $\mathcal{H}$.

Proof. Let Γ be the first ordinal with cardinality $\mathfrak{m}$. Well order $\mathcal{B}$ as $\mathcal{B} = \{B(\alpha) | \alpha < \Gamma\}$ and suppose each member of $\mathcal{B}$ meets at least $\mathfrak{m}$ members of $\mathcal{H}$. Inductively choose members $H(\alpha) \in \mathcal{H}$ for $1 \leq \alpha < \Gamma$ such that

(a) if $\alpha \neq \alpha'$ then $H(\alpha) \neq H(\alpha')$,

(b) $H(\alpha) \cap B(\alpha) \neq \emptyset$.

For each $\alpha < \Gamma$ choose a point $q(\alpha) \in H(\alpha) \cap B(\alpha)$; it is not required that $q(\alpha)$'s be distinct. Let $K(\alpha) = \{q(\alpha)\}$. Since $\mathcal{H}$ is hereditarily closure preserving the set $K = \bigcup \{K(\alpha) | \alpha < \Gamma\}$ must be closed, and K does not contain p. Yet each member of $\mathcal{B}$ meets K, so K cannot be closed unless it does contain p.

Corollary 2. *Let* p *be a point of a* T_1*-space* X. *Suppose* p *has a neighborhood base* $\mathcal{B}$ *whose cardinality is* $\mathfrak{m}$. *If* $\mathcal{H}$ *is any HCP collection such that for each* $H \in \mathcal{H}$, p *is not an isolated point of* H, *then some neighborhood of* p *meets fewer than* $\mathfrak{m}$ *members of* $\mathcal{H}$.

Proof. Let $\mathcal{H}' = \{H \setminus \{p\} | H \in \mathcal{H}\}$. Lemma 1 yields a neighborhood B of p meeting fewer than $\mathfrak{m}$ members of $\mathcal{H}'$. Since, for a set $H \in \mathcal{H}$, $B \cap (H \setminus \{p\}) \neq \emptyset$ if and only if $B \cap H \neq \emptyset$, we see that B meets fewer than $\mathfrak{m}$ members of $\mathcal{H}$.

Corollary 3. *Let* p *be a nonisolated point of a* T_1*-space* X *and let* $\mathcal{H}$ *be an HCP collection of open subsets of* X. *If* p *has a countable neighborhood base, then* $\mathcal{H}$ *is locally finite at* p.

Lemma 4. *Suppose* p *is a limit point*[3] *of a set* A *in a space* X *and that there is a* G_δ*-subset* G *of* X *which contains* p *and has* $G \cap (A \setminus \{p\}) = \emptyset$. *Then any HCP collection of neighborhoods of* p *must be finite.*

Proof. Write $G = \bigcap \{G(n) | n \geq 1\}$ where each $G(n)$ is an open subset of X. Suppose $\mathcal{C}$ is an infinite HCP collection of neighborhoods of p. Let $C(1), C(2), \ldots$ be distinct members of $\mathcal{C}$. Define $D(1) = A \cap C(1) \cap G(1)$ and $D(n) = D(n-1) \cap C(n) \cap G(n)$ whenever $n \geq 2$. Because $C(1) \cap G(1)$ is a neighborhood of p, p is a limit point of $D(1) \setminus \{p\}$. However p is not a limit

---

[3] I.e., $p \in \mathrm{cl}(A \setminus \{p\})$.

point of any set $D(n)\setminus D(n+1)$ so that because the $C(n)$'s are distinct members of an HCP collection, p cannot be a limit point of the set $\bigcup\{D(n)\setminus D(n+1)|n \geq 1\} = D_1\setminus\{p\}$. That contradiction establishes the lemma.

Theorem 5. *A regular space X is metrizable if and only if X has a σ-HCP base of open sets.*

Proof. That every metrizable space has such a base follows directly from the Nagata-Smirnov theorem.

To prove the converse assertion, let $\mathcal{B} = \bigcup\{\mathcal{B}(n)|n \geq 1\}$ be a σ-HCP base for X. Let p be a nonisolated point of X. Then $\{p\}$ is a G_δ-subset of X. For each fixed m the collection $\{B \in \mathcal{B}(m)|p \in B\}$ is finite, in light of Lemma 4, so that p belongs to only countably many members of $\mathcal{B}$. Thus X is first-countable at p.

Since X is first-countable, it follows from Corollary 3 that each set $X(n) = \{x \in X|\mathcal{B}(n)$ is locally finite at $x\}$ contains all nonisolated points of X. Also, each $X(n)$ is an open set. Let $\mathcal{B}'(n) = \{B \cap X(n)|B \in \mathcal{B}(n)\}$. Then each collection $\mathcal{B}'(n)$ is locally finite at all points of X and the collection $\mathcal{B}' = \bigcup\{\mathcal{B}'(n)|n \geq 1\}$ contains a neighborhood base at each nonisolated point of X.

Let $\mathcal{B}''(n) = \{\{x\}|\{x\} \in \mathcal{B}(n)\}$. Each $\mathcal{B}''(n)$ is a discrete collection in X so that the collection $\bigcup\{\mathcal{B}'(n) \cup \mathcal{B}''(n)|n \geq 1\}$ is a σ-locally finite base for X. According to the Nagata-Smirnov theorem, X is metrizable.

A more subtle application of Lemma 4 yields a result on σ-HCP local bases at a point.

Theorem 6. *Suppose p is a nonisolated point of a T_1-space X and suppose $\bigcup\{\mathcal{B}(n)|n \geq 1\}$ is a σ-HCP base of neighborhoods of p. Then each $\mathcal{B}(n)$ is finite and X is first countable at p.*

Proof. For each $B \in \mathcal{B}(n)$ choose a point $y(B) \in B\setminus\{p\}$. Since $\mathcal{B}(n)$ is HCP, the set $F(n) = \{y(B)|B \in \mathcal{B}(n)\}$ is a closed set. Furthermore p is a limit point of the set $F = \bigcup\{F(n)|n \geq 1\}$. Let $G = X\setminus F$. Then G is a G_δ subset of X, $p \in G$, and $G \cap (F\setminus\{p\}) = \emptyset$. According to Lemma 4, each collection $\mathcal{B}(n)$ must be finite.

In an attempt to simplify the proofs of Theorems 5 and 6—by eliminating the need for the technical Lemma 4—the authors were led to the conjecture that if $\mathcal{H}$ is an open HCP collection in a space X, then $\bigcap\mathcal{H}$ is open in X. Unfortunately, as Example 8 will show, the conjecture is false; however we can prove

Proposition 7. *If* $\mathcal{H}$ *is an open HCP collection in a Hausdorff k-space* [3] X, *then* $\bigcap\mathcal{H}$ *is open.*[4]

Proof. One shows that for each (countably) compact set $K \subseteq X$, the collection $\{H \cap K | H \in \mathcal{H}\}$ contains only finitely many distinct subsets of K. Hence $K \cap (\bigcap\mathcal{H}) = \bigcap\{H \cap K | H \in \mathcal{H}\}$ is the intersection of finitely many relatively open subsets of K, so $K \cap (\bigcap\mathcal{H})$ is relatively open in K for each compact subset K of X. Since X is a k-space, $\bigcap\mathcal{H}$ is open in X.

Example 8. There is an open HCP collection $\mathcal{H}$ in a hereditarily paracompact space X such that $\bigcap\mathcal{H}$ is not open.

To construct X we need a definition and a technical lemma. Let us say that a function f: $[0, \omega_1[\to \mathcal{P}([0, \omega_1[)$[5], where ω_1 is the first uncountable ordinal, is *admissible* if for each $\alpha < \omega_1$, $f(\alpha)$ is a countable subset of $]\alpha, \omega_1[$.

Lemma. *Let* f: $[0, \omega_1[\to \mathcal{P}([0, \omega_1[)$ *be admissible. Then* $[1, \omega_1[\setminus \bigcup\{f(\alpha): 0 \leq \alpha < \omega_1\}$ *is uncountable.*

Proof. If $[1, \omega_1[\setminus \bigcup\{f(\alpha): 0 \leq \alpha < \omega_1\}$ were countable we could obtain an admissible function g such that $\bigcup\{g(\alpha) | 0 \leq \alpha < \omega_1\} = [1, \omega_1[$. Defining $\phi(0) = 0$ and $\phi(\beta) = \inf\{\alpha: \beta \in g(\alpha)\}$ if $1 \leq \beta < \omega_1$, we obtain a function ϕ: $[0, \omega_1[\to [0, \omega_1[$ such that $\phi(\beta) < \beta$ if $0 < \beta < \omega_1$ and such that $\phi^{-1}(\alpha)$ is countable for each $\alpha < \omega_1$. But according to a theorem of Alexandroff and Urysohn [2, p. 79], no such function can exist.

It is clear that if f and g are admissible then so is h, defined by $h(\alpha) = f(\alpha) \cup g(\alpha)$ for each $\alpha < \omega_1$. For each admissible f, let $N_f = [1, \omega_1] \setminus \bigcup\{f(\alpha): 0 \leq \alpha < \omega_1\}$ and let $\mathcal{N} = \{N_f: f \text{ is admissible}\}$. Then $\mathcal{N}$ is closed under finite intersections so that we may topologize the set $X = [0, \omega_1]$ by taking $\mathcal{N}$ to be a neighborhood base at ω_1 and by making each other point isolated. With that topology X is a hereditarily paracompact Hausdorff space.

---

[4] One can also prove that if $\mathcal{H}$ is an open HCP collection in a locally connected regular space then $\bigcap\mathcal{H}$ is open, the key lemma being that in any T_1-space the intersection of an open, countable, HCP collection is open. Given that lemma, suppose $x \in \bigcap\mathcal{H}$ where $\mathcal{H}$ is an open HCP collection in X. If $\mathcal{H}$ were infinite we could choose distinct members H_n of $\mathcal{H}$ and (using regularity and local connectedness of X) connected open sets $U_n \subset H_n$ in such a way that $x \in U_n$ and $\mathrm{cl}(U_{n+1}) \subsetneq U_n \cap H_{n+1}$ for each n. But then the set $\bigcap\{U_n: n \geq 1\}$ would be a nonempty, closed-and-open subset of X which is properly contained in the connected set U_1. That being impossible, $\mathcal{H}$ is finite and our assertion is established.

[5] $\mathcal{P}([0, \omega_1[)$ denotes the power set of $[0, \omega_1[$.

For each $\alpha < \omega_1$, let $H_\alpha =]\alpha, \omega_1]$ and let $\mathcal{H} = \{H_\alpha: 0 \leq \alpha < \omega_1\}$. Then $\mathcal{H}$ is a collection of open sets in X and $\bigcap \mathcal{H} = \{\omega_1\}$ is not open. To complete the example, we show that $\mathcal{H}$ is HCP. Suppose that $S_\alpha \subseteq H_\alpha$ for each $\alpha < \omega_1$; it will be enough to show that if $\omega_1 \notin \mathrm{cl}(S_\alpha)$ for each α, then $\omega_1 \notin \mathrm{cl}(\bigcup\{S_\alpha: \alpha < \omega_1\})$. Because $\omega_1 \notin \mathrm{cl}(S_\alpha)$ there is an admissible function f_α having $S_\alpha \subseteq \bigcup\{f_\alpha(\beta): 0 \leq \beta < \omega_1\}$. By modifying f_α if necessary, we may assume that $f_\alpha(\beta) = \emptyset$ whenever $\beta < \alpha$. Defining $g(\beta) = \bigcup\{f_\alpha(\beta): 0 \leq \alpha \leq \beta\}$ for $\beta < \omega_1$, we obtain an admissible function having $\bigcup\{S_\alpha: 0 \leq \alpha < \omega_1\} \subseteq \bigcup\{g(\beta): 0 \leq \beta < \omega_1\}$ so that $\omega_1 \notin \mathrm{cl}(\bigcup\{S_\alpha: 0 \leq \alpha < \omega_1\})$ as required.

The referee has suggested a possible improvement in our metrization theorem. Suppose that one considers collections $\mathcal{H}$ in a space X with the property that if a point $x(H) \in H$ is chosen for each $H \in \mathcal{H}$ then the set $\{x(H): H \in \mathcal{H}\}$ is a closed discrete subspace of X; such collections might reasonably be called *weakly HCP*. Then is it true that a regular space is metrizable if it has a σ-weakly HCP base? The question has an affirmative answer provided only k-spaces are considered: the proof of Proposition 7 shows that a k-space having a σ-weakly HCP base must be first countable so that the proof of Theorem 5, beginning with the third paragraph, shows that X (if regular) is metrizable. However, our next example shows that if X is not assumed to be a k-space, then the suggested generalization of Theorem 5 is false.

Example 9. There is a nonmetrizable, hereditarily paracompact space which has a σ-weakly HCP base.

Proof. Let A be the set of all ordinals having cardinality less than $\aleph_{\omega_0}$. Let Z be the product space $\{0, 1\}^A$ and let $\overline{0}$ be the element of Z having $\overline{0}(\alpha) = 0$ for each $\alpha \in A$. Let X be the set $\{\overline{0}\} \cup \{z \in Z:$ the set $\{\alpha \in A: z(\alpha) = 0\}$ is finite$\}$. Topologize X by making each point of $X \setminus \{\overline{0}\}$ isolated and by taking basic neighborhoods of $\overline{0}$ to be all sets of the form $U \cap X$ where U is a basic neighborhood of $\overline{0}$ in the product space Z. Then X is hereditarily paracompact and nonmetrizable.

Let $\mathcal{B}'(n) = \{\{z\}: z \in X \setminus \{\overline{0}\}$ and $|\{\alpha \in A: z(\alpha) = 0\}| = n\}$. Then each $\mathcal{B}'(n)$ is a discrete collection in X. For each basic open neighborhood U of $\overline{0}$ in Z the set $R(U) = \{\alpha \in A: \pi_\alpha[U] = \{0\}\}$ is finite, where $\pi_\alpha: Z \to \{0, 1\}_\alpha$ denotes the projection. Let $\mathcal{B}''(n) = \{U \cap X: U$ is a basic neighborhood of $\overline{0}$ in Z and $R(U) \subseteq [0, \omega_n[\}$, where ω_n is the first ordinal of cardinality $\aleph_n$. Since each set $U \cap X$ in $\mathcal{B}''(n)$ is uniquely determined by the finite set $R(U)$ of $[0, \omega_n[$, $|\mathcal{B}''(n)| \leq \aleph_n$. In order to show that $\mathcal{B}''(n)$ is weakly HCP it will be sufficient to show that if a point $z_U \in (U \cap X) \setminus \{\overline{0}\}$ is chosen for each $U \cap X \in \mathcal{B}''(n)$, then $\overline{0} \notin \mathrm{cl}\{z_U\colon U \cap X \in \mathcal{B}''(n)\}$. For each of the chosen points z_U, the set $S(U) = \{\alpha \in A\colon z_U(\alpha) = 0\}$ is finite. Since $|\mathcal{B}''(n)| \leq \aleph_n$ the set $S = \bigcup\{S(U)\colon U \cap X \in \mathcal{B}''(n)\}$ has cardinality not exceeding $\aleph_n$ so that we may choose $\beta \in A \setminus S$. But then the neighborhood $X \cap \{z \in Z\colon z(\beta) = 0\}$ of $\overline{0}$ contains no point z_U so that $\overline{0} \notin \mathrm{cl}\{z_U\colon U \cap X \in \mathcal{B}''(n)\}$, as required.

Since the collection $\bigcup\{\mathcal{B}'(n) \cup \mathcal{B}''(n)\colon n \geq 1\}$ is a base for X, the proof is complete.

BIBLIOGRAPHY

1. J. Ceder, *Some generalization of metric spaces*, Pacific J. Math. 11 (1961), 105–125. MR 24 #A1707.

2. I. Juhász, *Cardinal functions in topology*, Math. Center Tracts, North-Holland, Amsterdam, 1971.

3. J. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955. MR 16, 1136.

4. J. Nagata, *On a necessary and sufficient condition of metrizability*, J. Inst. Polytech. Osaka City Univ. Ser A Math. 1 (1950), 93–100. MR 13, 264.

5. Ju. M. Smirnov, *A necessary and sufficient condition for metrizability of a topological space*, Dokl. Akad. Nauk SSSR 77 (1951), 197–200. (Russian) MR 12, 845.

DEPARTMENT OF MATHEMATICS, MIAMI UNIVERSITY, OXFORD, OHIO 45056

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PENNSYLVANIA 15260